Lecture 4: Modeling with Differential Equations.

Example 1:

Differential equations can model polar states;

- religion
- party affliation
- left/right handed
- language
- party affliation.

Let x denote fraction of population that is republican and $1-x$ denote the fraction of population that is democrat y.

Note:

$$\underbrace{x}_{rep} + \underbrace{(1-x)}_{dem \Rightarrow y} = 1 \longrightarrow \text{total population.}$$

$$\frac{dx}{dt} = -\underbrace{P_{xy}}_{\text{frac. that switch to dem}} x + \underbrace{P_{yx}}_{\text{frac. that switch to rep.}} (1-x)$$

$P_{yx} = s x^{a}$, and $P_{xy} = (1-s)(1-x)^{a}$

s: Status (i.e. how awesome the party is)

$a > 1$, some parameter. This function ~~measures~~ quantifies that a party is more popular for larger number of members.

$(1-s)$: this party is less attractive if $s > 1/2$

$(1-x)^a$: same population attractive function.

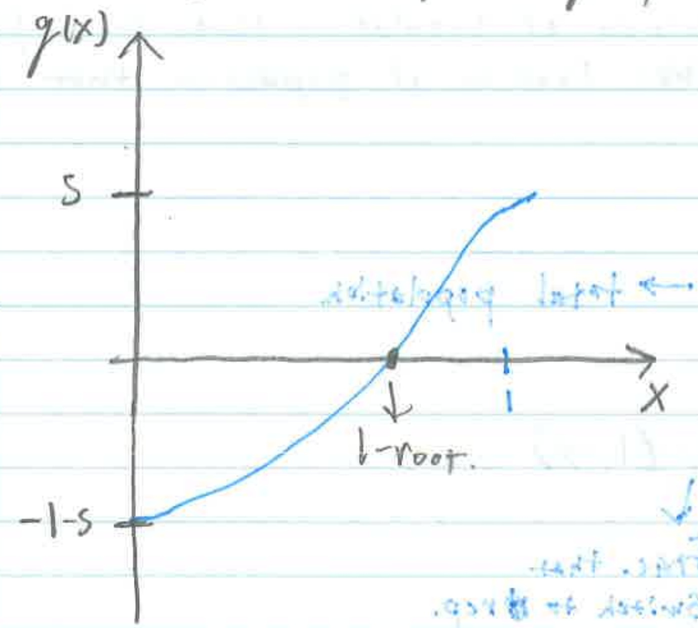
Lets analyze what this model predicts

$$\frac{dx}{dt} = s x^a (1-x) - (1-s)(1-x)^a x$$

$$= x(1-x)\underbrace{\left(s x^{a-1} - (1-s)(1-x)^{a-1}\right)}_{g(x)}$$

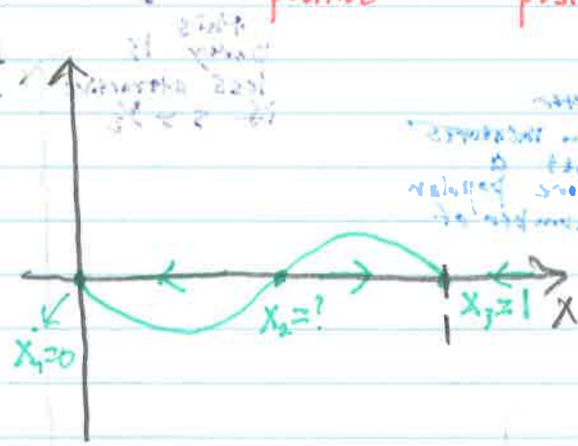
$$= x(1-x)\,g(x) = f(x).$$

Lets try sketching a graph:



Note:

$$g'(x) = s(a-1)\underbrace{x^{a-2}}_{\text{positive}} + (1-s)\underbrace{(1-x)}_{\text{positive}}^{a-2}\ a-1 > 0$$



This model predicts that the population will <u>all become democrats or republicans</u>.

Stability analyzed via:

$$f'(x) = a s x^{a-1}(1-x) - s x^{a+1} + (1-s)(1-x)^{a-1} a x - (1-s)(1-x)^a$$

$$\Rightarrow f'(0) = -(1-s),$$

$$f'(1) = -s,$$

Hence, 0, and 1 are unstable $\Rightarrow X_2$ has to be stable.

Example 2:

Modeling spread of infectious diseases

S - population that is susceptible
I - population that is infected

Modeling:

- When a susceptible encounters an infected there is a chance the susceptible becomes infected.
- Infected can recover and return to being susceptible
- Infected can die.

$$\frac{dS}{dt} = -\underbrace{\alpha I S}_{\text{infection}} + \underbrace{\beta I}_{\text{recovering from infection}}$$

$$\frac{dI}{dt} = \underbrace{\alpha I S}_{\text{infection}} - \underbrace{\beta I}_{\text{recovering}} - \underbrace{\gamma I}_{\text{dying}}$$

$$\frac{dR}{dt} = \underbrace{\gamma I}_{\text{dying}}$$

For now, assume $\gamma = 0$ (Nobody dies)

$$\Rightarrow \frac{dS}{dt} = -\alpha I S + \beta I$$

$$\frac{dI}{dt} = \alpha I S - \beta I.$$

Now,

$$\frac{dI}{dt} + \frac{dS}{dt} = 0$$

The total population is $P = I + S$. We have found

$$\frac{d}{dt} P = 0.$$ (Total population remains the same)

It follows that

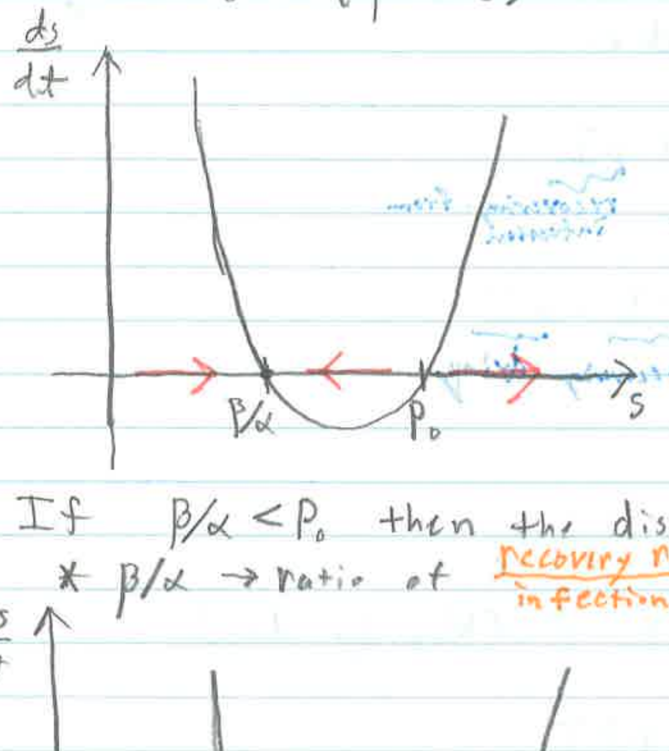
$$I + S = P_0$$

$$\Rightarrow I = P_0 - S$$

The equation for S becomes:

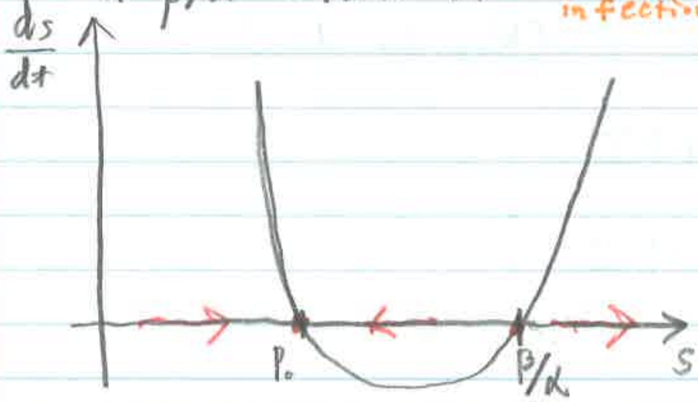
$$\frac{dS}{dt} = -\alpha(P_0 - S)S + \beta(P_0 - S)$$

$$= (P_0 - S)(\beta - \alpha S)$$



If $\beta/\alpha < P_0$ then the disease remains endemic

* $\beta/\alpha \rightarrow$ ratio of $\frac{\text{recovery rate}}{\text{infection rate}}$.



*If $\beta/\alpha > P_0$ then everybody recovers.

Example 3:

$$\frac{dN}{dt} = rN - aN(N-b)^2$$

$$= rN - ab^2 N(N/b - 1)^2$$

We assume that this models a population.

rN - exponential growth

$-aN(N-b)$ - term that penalizes growth for small and large values of N.

Dimensional Analysis

Variables:

$[N]$ - population

$[t]$ - time

$\left[\frac{dN}{dt}\right]$ - $\frac{\text{pop.}}{\text{time}}$

Paramaters:

$[r]$ - time^{-1}

$[b]$ - population

$[a]$ - $\frac{\text{population}^{-2}}{\text{time}}$

Change of Variables:

$x = \frac{N}{\alpha}$, $[\alpha]$ - population

$\tau = \frac{t}{\beta}$, $[\beta]$ - time

$\Rightarrow N = \alpha x$.

$$\Rightarrow \frac{dN}{dt} = \frac{d}{dt}(\alpha x) = \frac{d\tau}{dt}\frac{d}{d\tau}(\alpha x) = \frac{\alpha}{\beta}\frac{dx}{d\tau}$$

Set $\alpha = b$, $\beta = r^{-1}$.

$$\Rightarrow br\frac{dx}{d\tau} = rbx - ab^3x(x-1)^2$$

$$\Rightarrow \frac{dx}{d\tau} = x - \frac{ab^2}{r}x(x-1)^2$$

Set $\gamma = ab^2/r$ we have that:

$$\frac{dx}{d\tau} = x - \gamma x(x-1)^2$$

Analysis

Fixed points:

$x=0$, $1-\gamma(x-1)^2 = 0$

$$\Rightarrow \frac{\pm 1}{\sqrt{\gamma}} = x-1$$

$$\Rightarrow x = 1 \pm \frac{1}{\sqrt{\gamma}}$$

We have three fixed points:

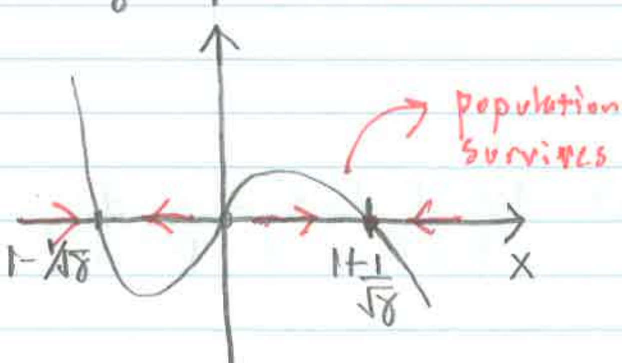
We want to determine when $1 - 1/\sqrt{\gamma} > 0$

$$\Rightarrow \sqrt{\gamma} - 1 > 0$$

$$\Rightarrow \gamma > 1.$$

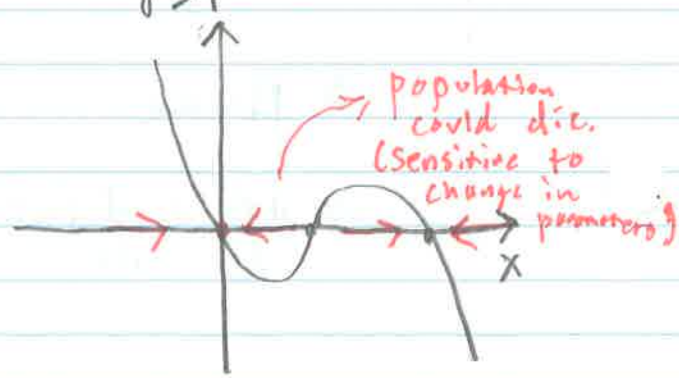
Case 1:

$\gamma < 1$

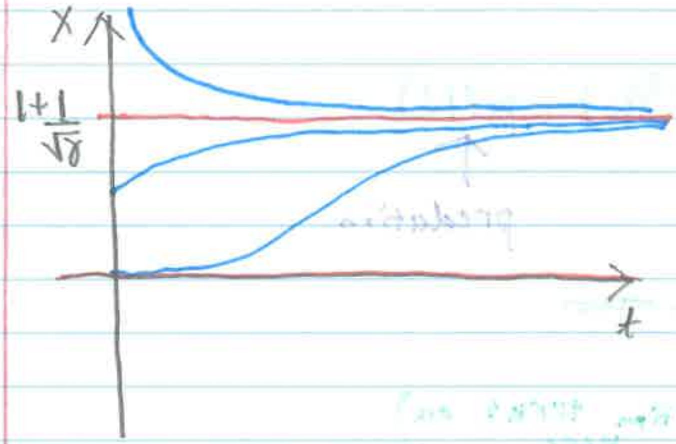


Case 2:

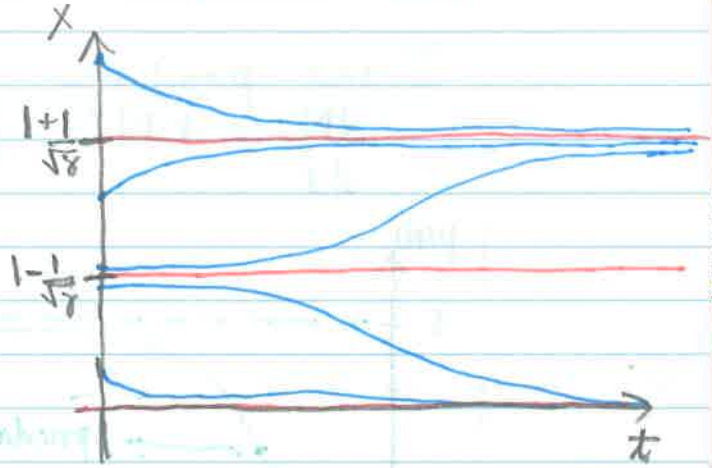
$\gamma > 1$



Solution curves:

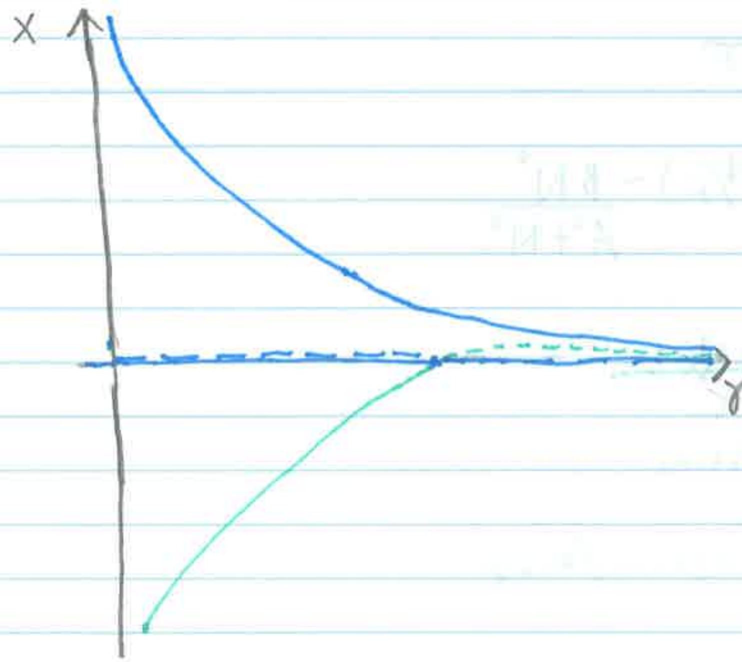


Case 1: Logistic



Case 2: Logistic or extinction.

Bifurcation curve:



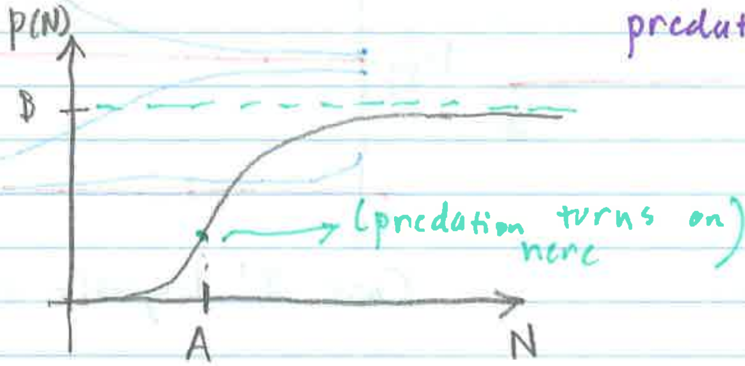
Plot fixed points vs parameter. Indicate stability by solid lines and unstable fixed points by dashed curves.

Example 4: Insect Outbreak

Insect population

$$\frac{dN}{dt} = rN(1 - N/K) - p(N)$$

predation



$$p(N) = \frac{BN^2}{A^2+N^2}$$

$$\frac{dN}{dt} = rN(1 - N/K) - \frac{BN^2}{A^2+N^2}$$

Dimensional Analysis

Variables

$[N]$ - population

$[t]$ - time

$[\frac{dN}{dt}]$ - population/time

parameters:

$[r]$ - $time^{-1}$

$[K]$ - population

$[B]$ - $time^{-1}$ $population^{-1}$

$[A]$ - $population^{-1}$

This motivates rescaling by

$$x = \frac{N}{A}, \quad \tau = \frac{B}{A} t$$

$$\Rightarrow N = Ax, \quad \frac{dN}{dt} = \frac{d}{dt} Ax = B\frac{dx}{d\tau}$$

Therefore, the system reduces to:

$$\frac{dx}{dt} = \alpha x(1 - x/\beta) - \frac{x^2}{1+x^2} = f(x).$$

Fixed Points.

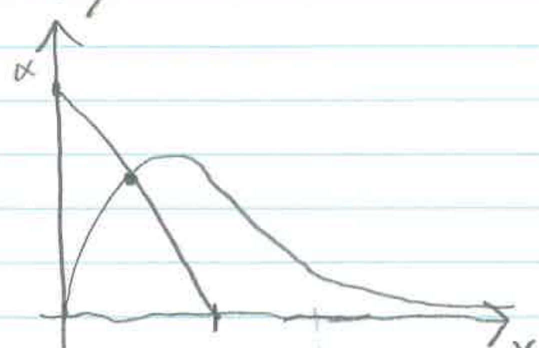
One obvious fixed point is $x = 0$. Calculating we have that

$$f'(x) = \alpha - \frac{2\alpha x}{\beta} - \frac{(1+x^2)2x - x^2(2x)}{(1+x)^2}$$

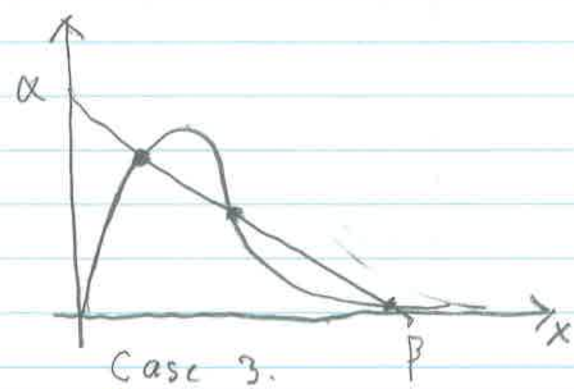
$$\Rightarrow f'(0) = \alpha.$$

Consequently, $x = 0$ is always unstable.

To determine the other fixed points we take a graphical approach. Let $g(x) = \alpha(1 - x/\beta) - x/1+x^2$. We try to analyze when $\alpha(1 - x/\beta) = x/1+x^2$. We do this graphically.



Case 1.



Case 3.

To determine when case 1 switchs to case 2 we look where the line passes tangentially to $\frac{x}{1+x^2}$. This gives us the conditions:

a.) $\frac{-\alpha}{\beta} = \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$,

b.) $\alpha x(1-x/\beta) = \frac{x^2}{1+x^2}$.