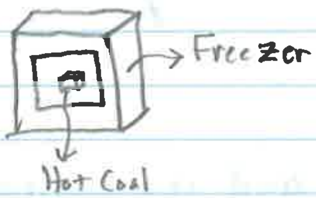
Lecture 3: First Order Differential Equations.

Differential equations model the flow, or rate of change over time of a physical process.

Example:

An object placed in a cold (hot) environment at a fixed temperature T_E.



If the temperature difference between the object and its environment changes at a rate proportional to the temperature difference, what is a function for the temperature of an object.

$$\frac{dT}{dt} = K(T_E - T) \quad \text{(Newton's Law of Cooling)}$$

conductance

temperature of environment

$$T(0) = T_0 \quad \text{(Initial Condition)}$$

Solution:

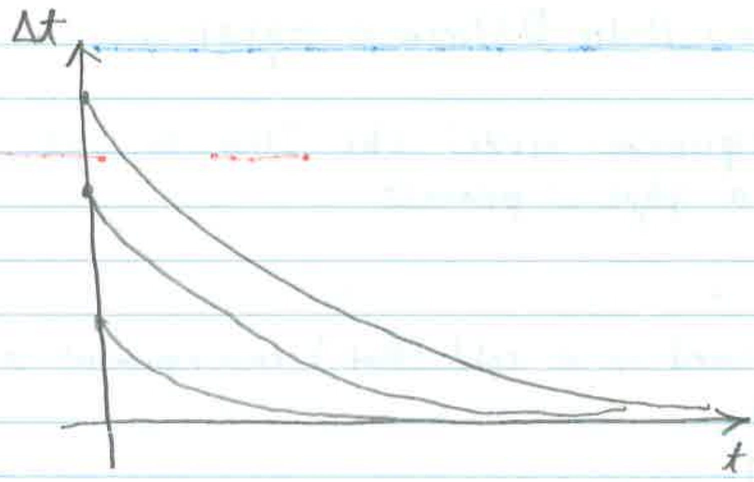
Let $\Delta T = T - T_E$, and $\Delta T_0 = T_0 - T_E$.

$$\Rightarrow \frac{dT}{dt} = \frac{d\Delta T}{dt}$$

$$\Rightarrow \frac{d\Delta T}{dt} = -K\Delta T$$

$$\Rightarrow \Delta T = \Delta T_0 \, e^{-Kt}$$

*Exact quantitative prediction of temperature.



Example:

Now suppose the object and its environment influence each other:

$$\frac{dT}{dt} = -K_1 (T - T_E) \qquad T(0) = T_0$$

$$\frac{dT_E}{dt} = K_2 (T - T_E) \qquad T_E(0) = T_{E,0}$$

Let $\Delta T = T - T_E$, $H = T + T_E$. Then,

$$\frac{dH}{dt} = (K_2 - K_1)\Delta T$$

$$\frac{d\Delta T}{dt} = -(K_1 + K_2)\Delta T$$

$$\Rightarrow \Delta T = \Delta T_0 \, e^{-(K_1+K_2)t}$$

$$\Rightarrow \frac{dH}{dt} = (K_2 - K_1) e^{-(K_1+K_2)t}$$

$$\Rightarrow H(t) = \frac{(K_1 - K_2)}{(K_1 + K_2)} \Delta T_0 \left(e^{-(K_1+K_2)t} - 1 \right) + H_0$$

Consequences:

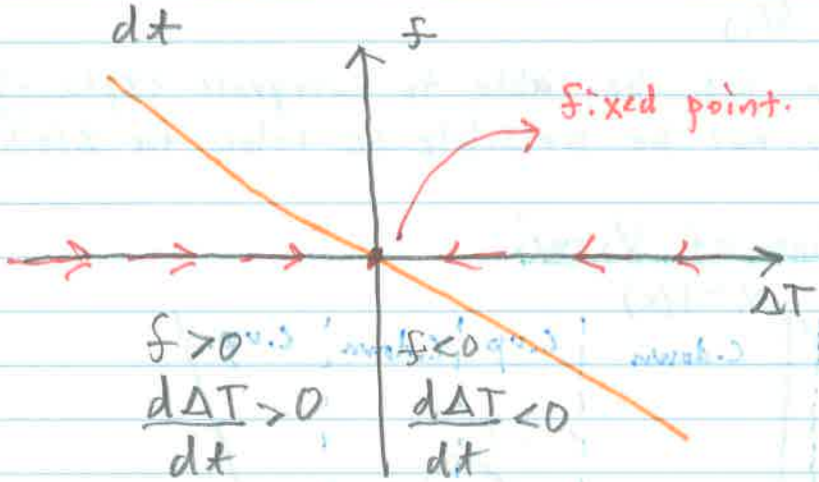
If $K_1 = K_2$ total heat is conserved:

$H(t) = H_0$

(Thermodynamic Principle)

Graphical Analysis:

$$\frac{d\Delta T}{dt} = -K\Delta T = f(\Delta T)$$



$\lim_{t\to\infty} \Delta T = 0$ (This is easy to see graphically).

For $\Delta T > 0$, ΔT is always decreasing and only stops flowing when $\Delta T = 0$.

"proof"

1. If $\lim_{t\to\infty} \Delta T = c$, then $\lim_{t\to\infty} \frac{d\Delta T}{dt} = 0$ but $\frac{d\Delta T}{dt} = 0$ if and only if $\Delta T = 0$.

2. $\lim_{t\to\infty} \Delta T \neq \pm\infty$.

General Framework:

$$\frac{dx}{dt} = f(x)$$

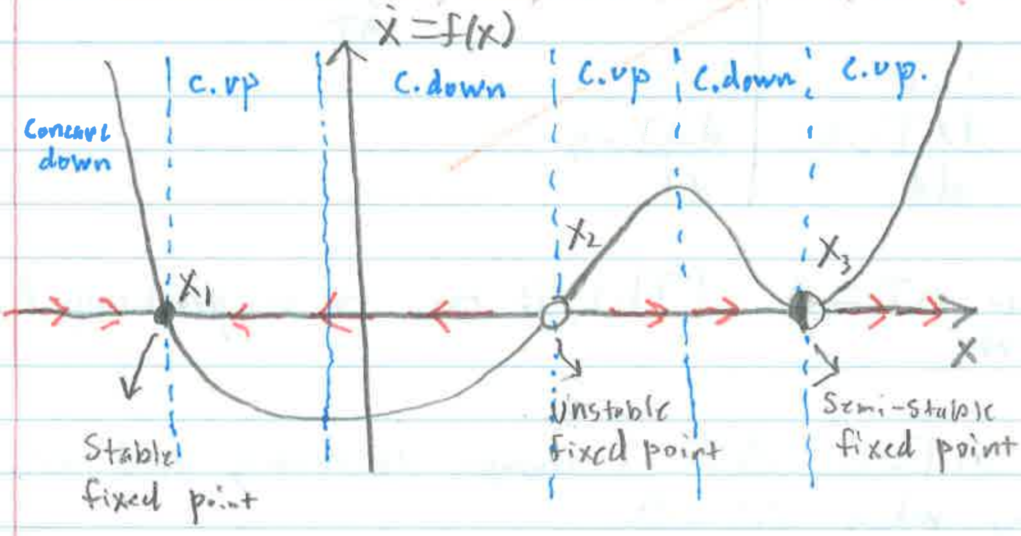
x – position
$f(x)$ – velocity

We can try to solve:

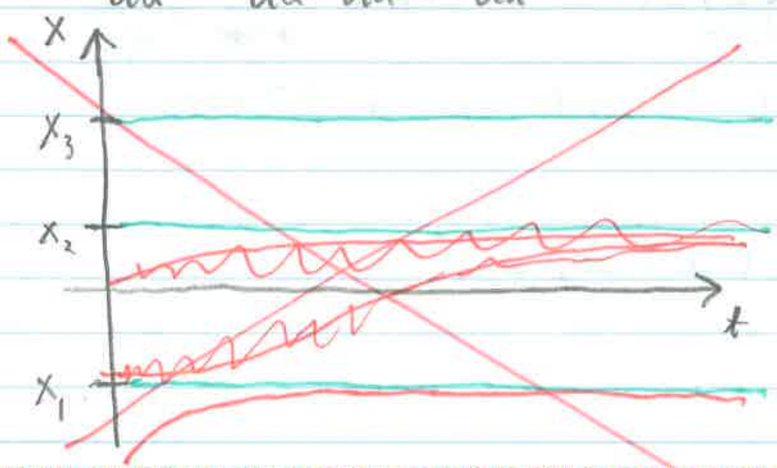
$$t = \int_{x_0}^{x} \frac{1}{f(s)}\, ds$$

i.) We may not be able to integrate explicitly.
ii) It may not be possible to solve for $x(t)$.

Geometric Point of View:

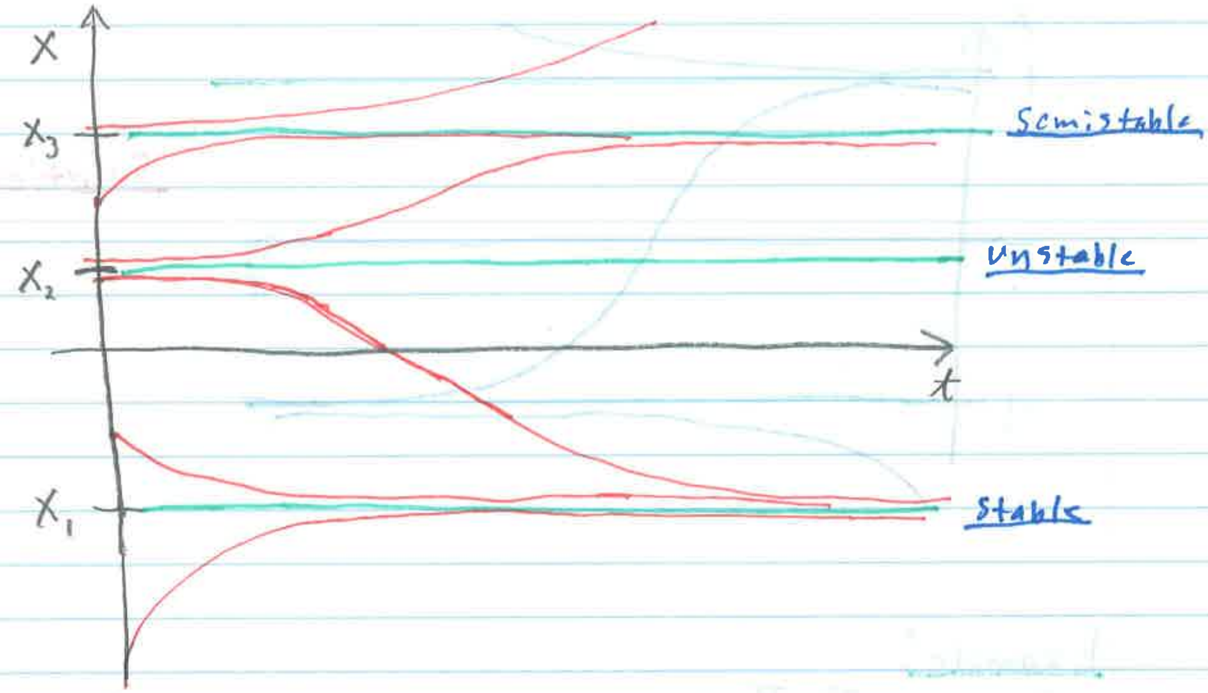


$$\frac{d^2x}{dt^2} = \frac{d}{dt}\frac{dx}{dt} = \frac{d}{dt} f(x) = f'(x)\cdot\frac{dx}{dt} = f'(x) f(x)$$



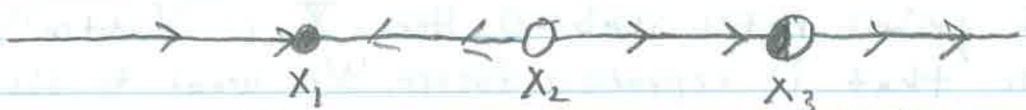
Ignore!!

→ Mr. Yuck.



General qualitative description of solution curves.

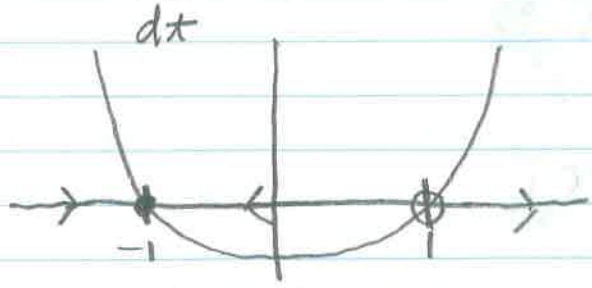
The phase portrait (vector field) captures all of this behavior.

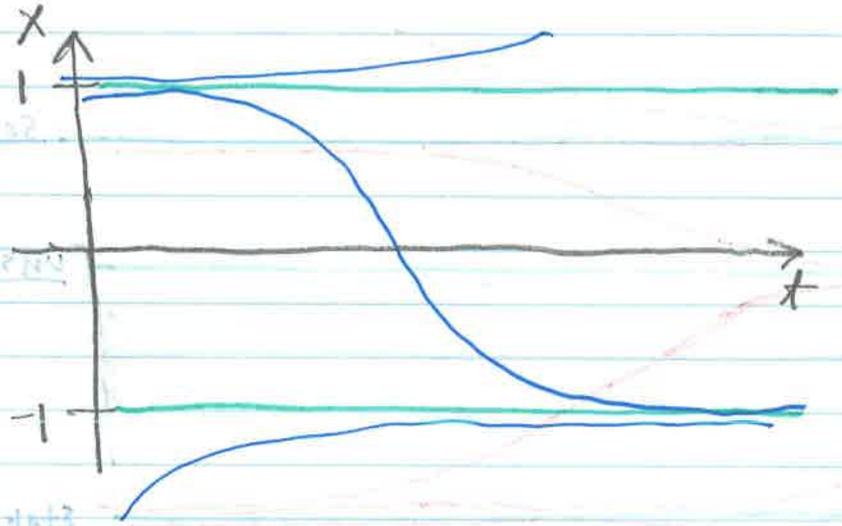


At each point on the real line, the function f assigns a direction pointing left or right. Stability can be determined graphically.

Example:

$$\frac{dx}{dt} = x^2 - 1$$





Solution Curves:

Example:

Suppose X, Y are two species that reproduce exponentially fast

$$\frac{dX}{dt} = aX$$

$$\frac{dY}{dt} = bY$$

With growth rates $a > b > 0$. Here X is "fitter" in the sense that it reproduces faster. We want to show that $X(t)$ dominates the population in the long run.

$$x(t) = \frac{X(t)}{X(t)+Y(t)} \quad \left(\begin{matrix}\text{ratio of } X \text{ population} \\ \text{to total population}\end{matrix}\right)$$

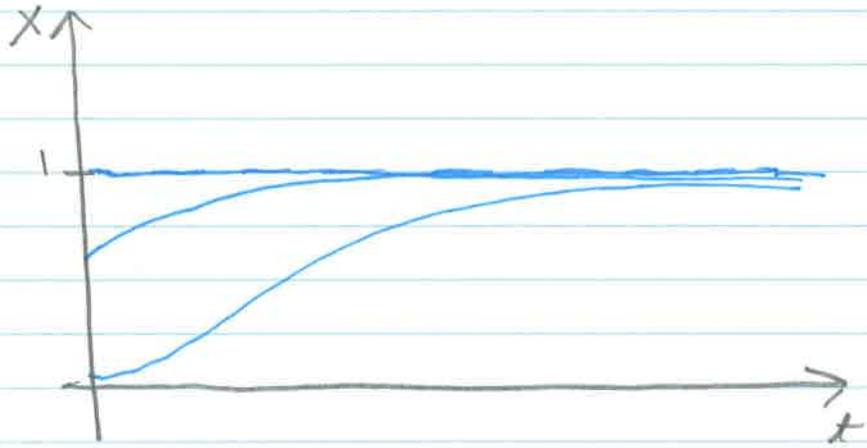
$$\frac{dx}{dt} = \frac{(X+Y)\frac{dX}{dt} - X\left(\frac{dX}{dt}+\frac{dY}{dt}\right)}{(X+Y)^2}$$

$$= \frac{Y\frac{dX}{dt} - X\frac{dY}{dt}}{(X+Y)^2}$$

$$= \frac{aXY - bXY}{(X+Y)^2}$$

$$\Rightarrow \frac{dx}{dt} = \frac{(a-b) X \cdot Y}{(X+Y)^2}$$

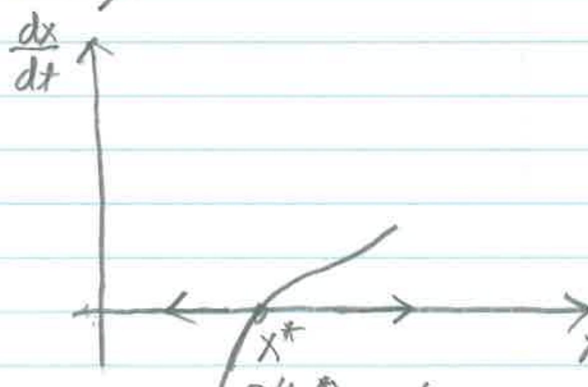
$$= (a-b)\, x (1-x) \quad (\text{Logistic Equation})$$



The population ratio always converges to dominance of x.

Analytical Approach to Stability

$f'(x^*) < 0$
x^* is stable.



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