Lecture 2: Discrete Dynamical Systems

One useful type of model we have discussed is discrete dynamical systems:

$$x_{n+1} = f(x_n)$$

x_{n+1}: new value x_n: old value

example

Logistic growth

$$x_{n+1} = r(K - x_n)x_n \rightarrow \text{let } y_n = x_n/K \Rightarrow y_{n+1} = \bar{r}(1-y_n)y_n$$

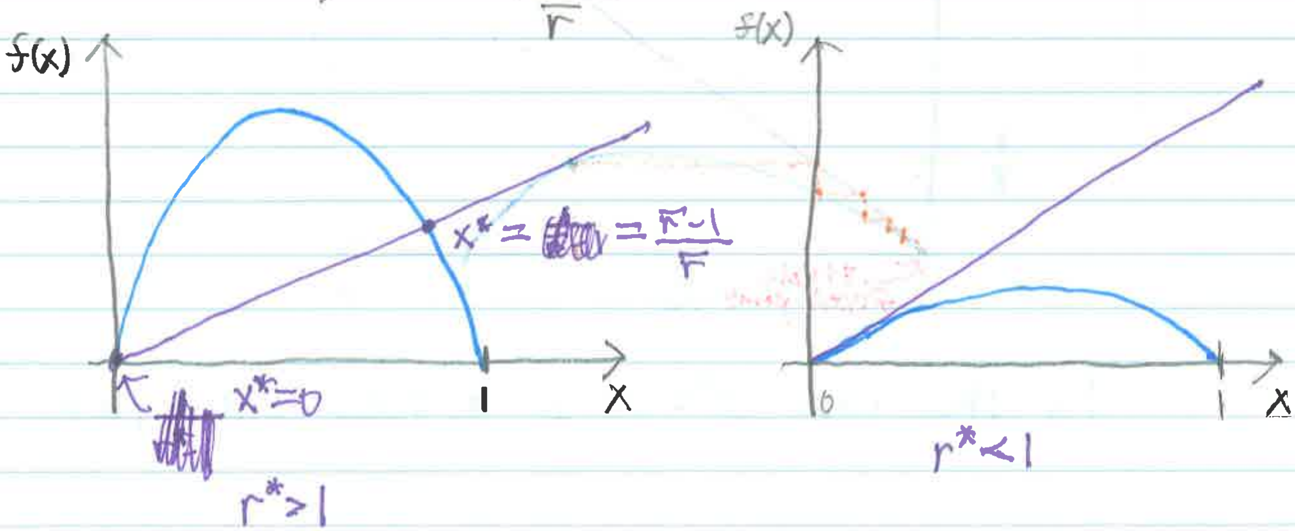
Fixed Points:

We say that x^* is a fixed point of a discrete dynamical system if:

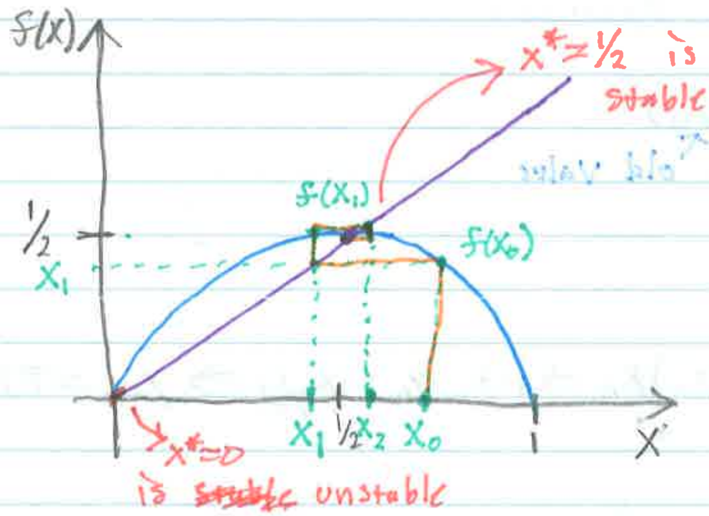
$$x^* = f(x^*)$$

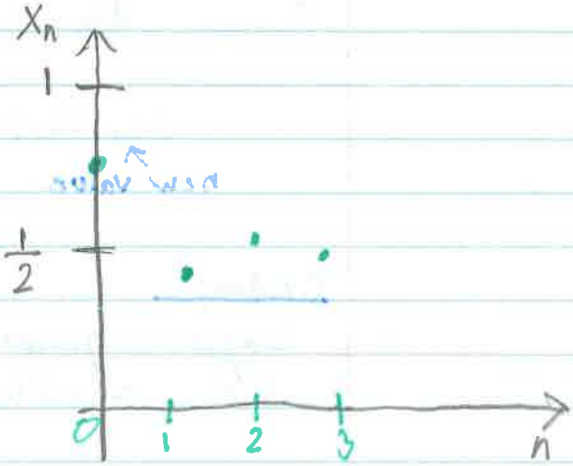
Lets determine fixed points of the logistic equation:

$$x^* = \bar{r}(1-x^*)x^*$$
$$\Rightarrow x^*(1 - \bar{r}(1-x^*)) = 0$$
$$\Rightarrow x^* = 0, \quad x^* = \frac{\bar{r}-1}{\bar{r}}$$



Lets simulate the growth of the population by hand for $r=2$.

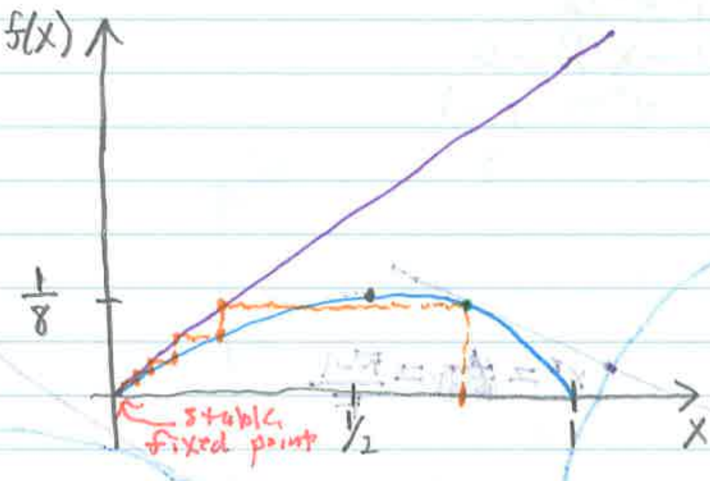




We say that a fixed point x^* is stable if nearby trajectories go to x^* as $n \to \infty$, i.e. $\lim_{n\to\infty} x_n = x^*$.

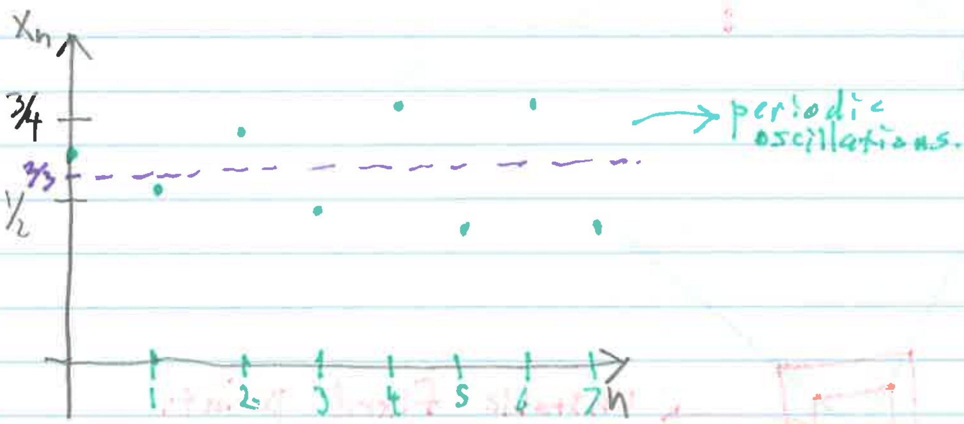
We say a fixed point is unstable if it is not stable.

What happens if $r = 1/2$



In this case $x^*=0$ is a stable fixed point.

What happens if $r=3$?



How can we test stability?

1. If $|f'(x^*)|<1$, x^* is stable (see next page)
2. If $|f'(x^*)|>1$, x^* is unstable

When does $x^*=\frac{r-1}{r}$ become unstable?

$f(x) = r(1-x)x$

$\Rightarrow f'(x) = r(1-2x)$

$\Rightarrow |f'(x^*)| = \left|r\left(1-\frac{2(r-1)}{r}\right)\right| < 1$

$\Rightarrow |r(r-2r+2)| < r$

$|r(2-r)| < r$

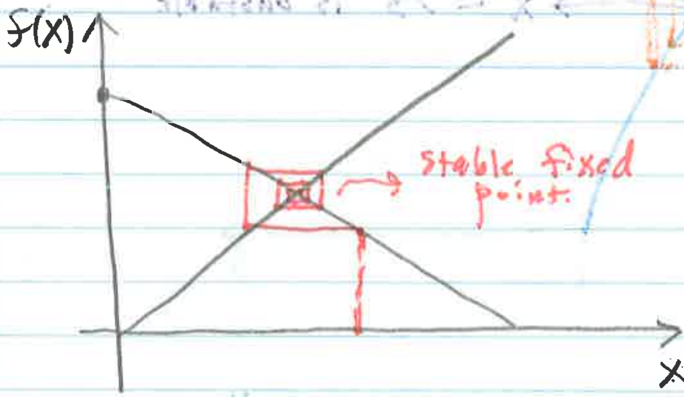
$\Rightarrow r(2-r)<r$ and $(r-2)r<r$

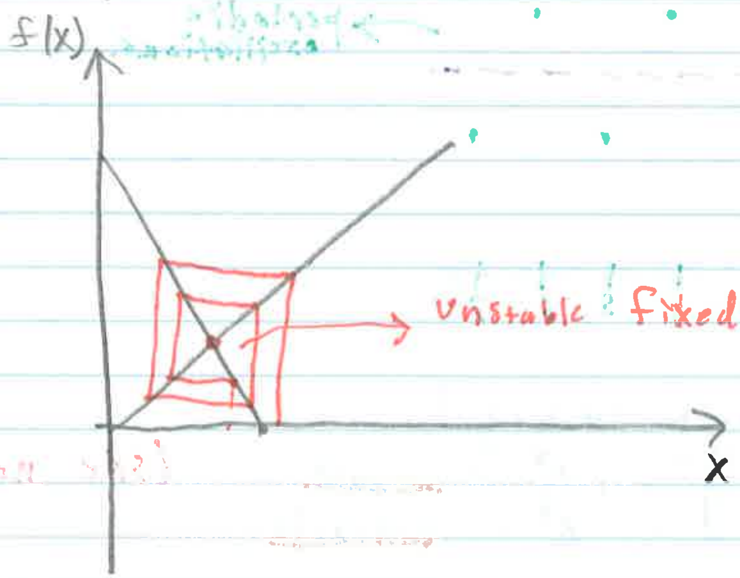
$\Rightarrow r>1$, and $r<3$.

Idea behind stability theorem

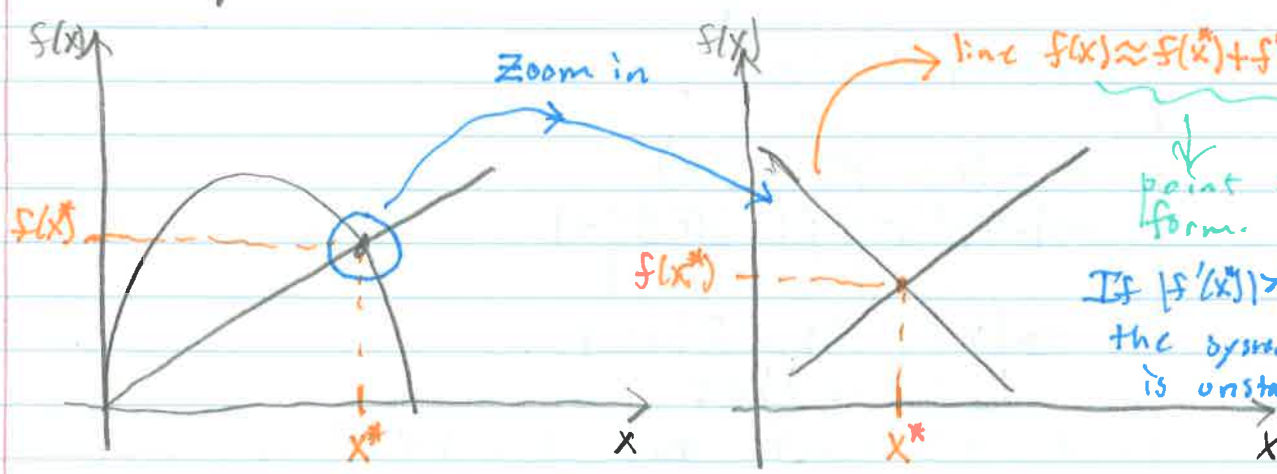
Lets look at linear systems

$$X_{n+1} = -mX_n + b = f(x)$$





For a general system

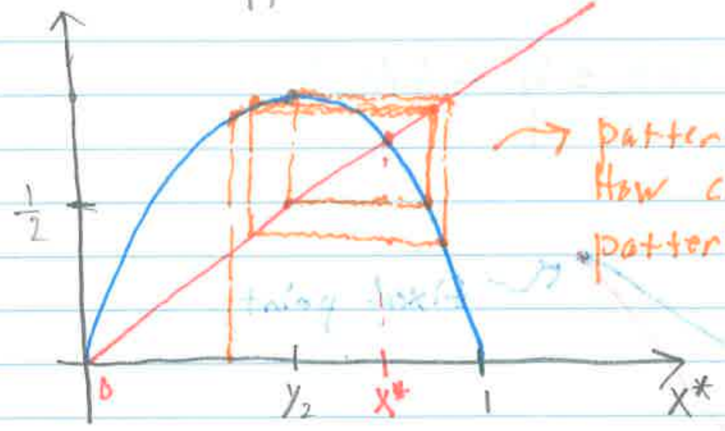


line $f(x) \approx f(x^*) + f'(x^*)(x - x^*)$

point slope form.

If $|f'(x^*)| > 1$ the system is unstable.

What happens if $r > 3$?



pattern repeats. How can we find repeat patterns?

Analyze:

$$x_{n+2} = f(x_{n+1}) = f(f(x_n))$$

Periodic orbit with period 2 satisfy:

$$x^* = f(f(x^*))$$
$$x^* = f(r(1-x^*)x^*)$$
$$x^* = r(1 - r(1-x^*)x^*)r(1-x^*)x^*$$

We have to solve this quantic equation

$$\Downarrow$$

$$x^*(r(1-r(1-x^*)x^*)r(1-x^*)-1) = 0$$
$$x^*(r^2(1-rx^*+rx^{*2})(1-x^*)-1) = 0$$
$$x^*(r^2(1-rx^*+rx^{*2}-x^*+rx^{*2}-rx^{*3})-1) = 0$$
$$x^*(r^3x^{*3}-(1+r)r^2x^*+(r^2-1)) = 0$$

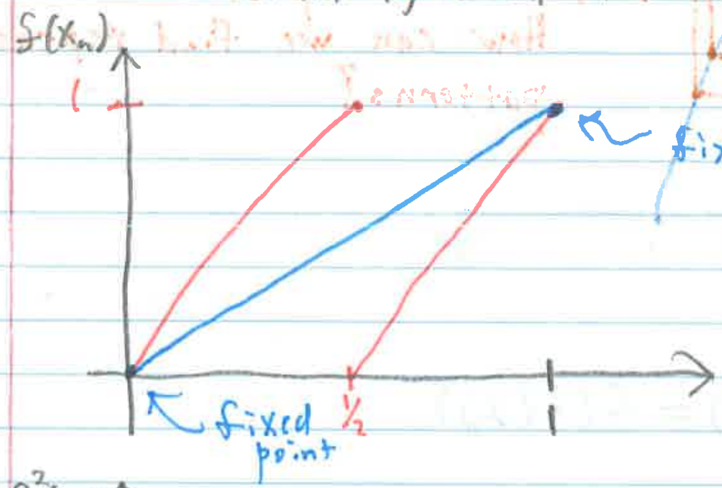
Roots:

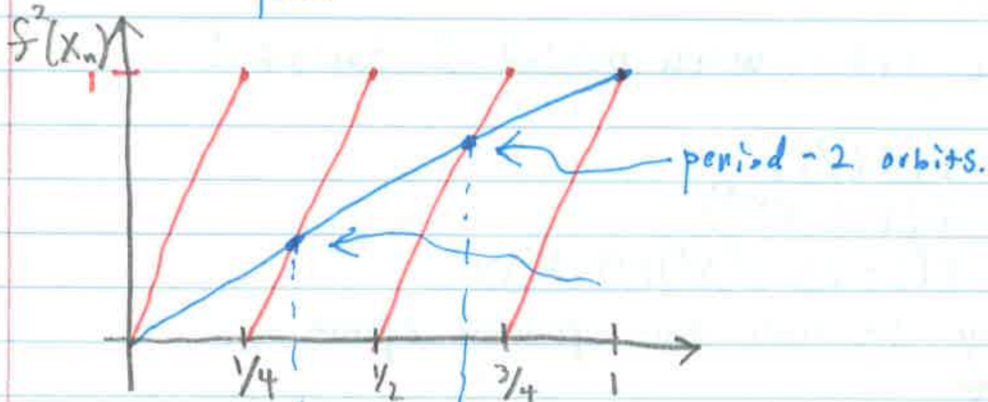
$$x^* = \frac{r-1}{r}, \quad x^* = 0$$

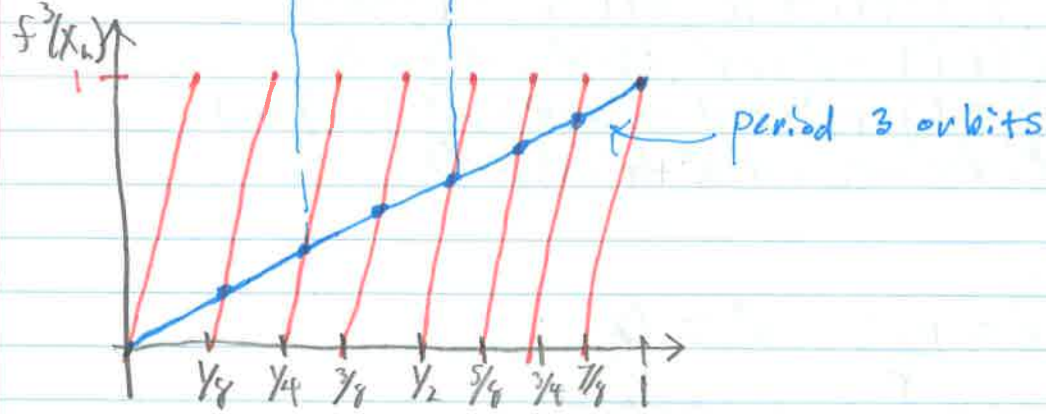
$$x^* = \frac{r+1}{2r} \pm \frac{\sqrt{(r-3)(r+1)}}{2r}$$

Interesting Example:

$$x_{n+1} = \begin{cases} 2x_n, & 0 \le x_n \le \tfrac{1}{2} \\ 2x_n - 1, & \tfrac{1}{2} \le x_n \le 1 \end{cases} = f(x_n)$$







In binary the decimal expansion

.101101101 → .01101101

shifts decimal to the write and removes largest digit.

All fixed points and periodic orbits are unstable.