Introduction to Mathematical Modeling

Lecture #1: What is Modeling

Model - Something that mimics relevant features of a situation being studied. (vague)

examples:

1. Road map of the United States
2. Aquarium

Mathematical Model - Models that mimic reality using the language of mathematics.

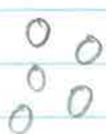
* A mathematical model is an abstract, simplified, mathematical construct related to a part of reality and created for a specific purpose.

examples:

1. Newton's theory of gravitation
2. Model of nuclear warfare
3. Models of the weather.

An Example

How can we model the growth of a population? Suppose we start with N_0 species



N_0

Can we write an equation for N_1, the number of species after 1 relative time step?

Model 1:

$$N_1 = N_0 + r N_0 \Delta t$$

N_1 → New population; N_0 → old population; r → growth rate; Δt → time step (1 minute, 1 day, 1 year etc.).

r - parameter (fixed constant) growth rate
N_n - population after n time steps:

$$N_{n+1} = N_n + \beta N_n \quad (\beta = r\Delta t)$$

$$\Rightarrow N_{n+1} = (1+\beta) N_n$$
$$= (1+\beta)^2 N_{n-1}$$
$$\vdots$$
$$= (1+\beta)^n N_0$$

(population grows exponentialy)

Model 2 (Discrete Logistic Equation)

The previous model allows for a population to grow without bound. In the short term this may be appropriate. What about in the long term?

$$N_{n+1} = N_n + r(N_n) N_n \Delta t$$

$r(N_n)$ → Growth rate is now a function of population.

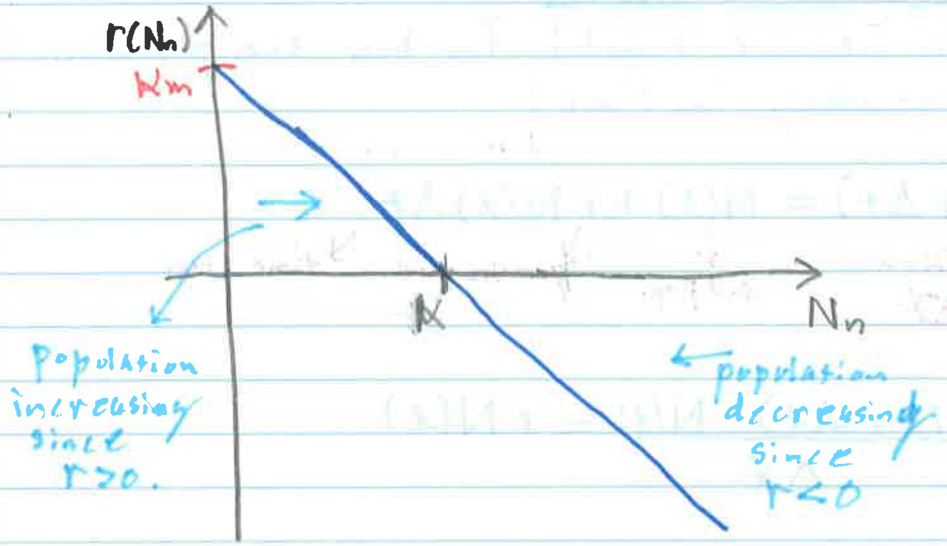
*Note, r could also

a.) If N is too large, $r(N)$ should be negative so that the population decreases.

b.) If N is small the population should rapidly increase due to abundant resources.

Lets make the easiest guess:

$$r(N_n) = -m(N_n - K)$$



$$N_{n+1} = N_n + \beta(K - N_n)N_n$$

$$= N_{n-1} + \beta(K - N_{n-1})N_{n-1} + \beta\left(K - N_{n-1} - \beta(K - N_{n-1})N_{n-1}\right)\left(N_{n-1} + \beta(K - N_{n-1})N_{n-1}\right)$$

$$= \left(N_{n-1} + \beta(K - N_{n-1})N_{n-1}\right)\left(1 + \beta\left(K - N_{n-1} - \beta(K - N_{n-1})N_{n-1}\right)\right)$$

$$= \left(N_{n-1} + \beta(K - N_{n-1})N_{n-1}\right)\left(1 + \beta(K - N_{n-1})(1 - \beta N_{n-1})\right)$$

$$\vdots$$

This approach to analyzing the model may not be too useful... Instead, we can turn to simulations, i.e. use a computer to output useful information.

We need to input 3 parameters to analyze the model

1. N_0 – starting population
2. β – per unit time growth factor
3. K – carrying capacity (maximum population)

stable population.

Model 3 (Continuous Growth):

Can we formulate a model for how the population $N(t)$ grows continuously in time?

$$\underset{\substack{\text{population}\\ \text{at time}\\ t+\Delta t}}{N(t+\Delta t)} = \underset{\text{old pop.}}{N(t)} + \underset{\text{growth rate}}{r}\, N(t) \underset{\text{time step.}}{\Delta t}$$

$$\Rightarrow \frac{N(t+\Delta t)-N(t)}{\Delta t} = rN(t)$$

$$\Rightarrow \lim_{\Delta t \to 0} \frac{N(t+\Delta t)-N(t)}{\Delta t} = rN(t).$$

$$\Rightarrow \frac{dN}{dt} = rN(t)$$ (Differential Equation)

We can solve for N:

$$\int \frac{1}{N}\, dN = \int r\, dt$$ (seperation of variables)

$$\ln(N) + c = rt$$

$$\Downarrow$$

$$\ln(N) = rt - c$$

$$\Rightarrow N(t) = e^{-c} e^{rt}$$

Now, $N(0) = N_0 \Rightarrow e^{-c} = N_0$. Therefore,

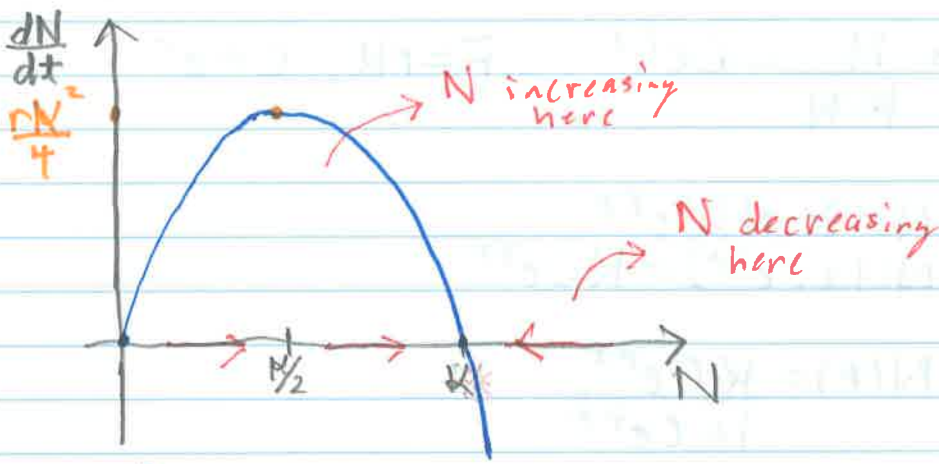
$$N(t) = N_0 e^{rt}.$$

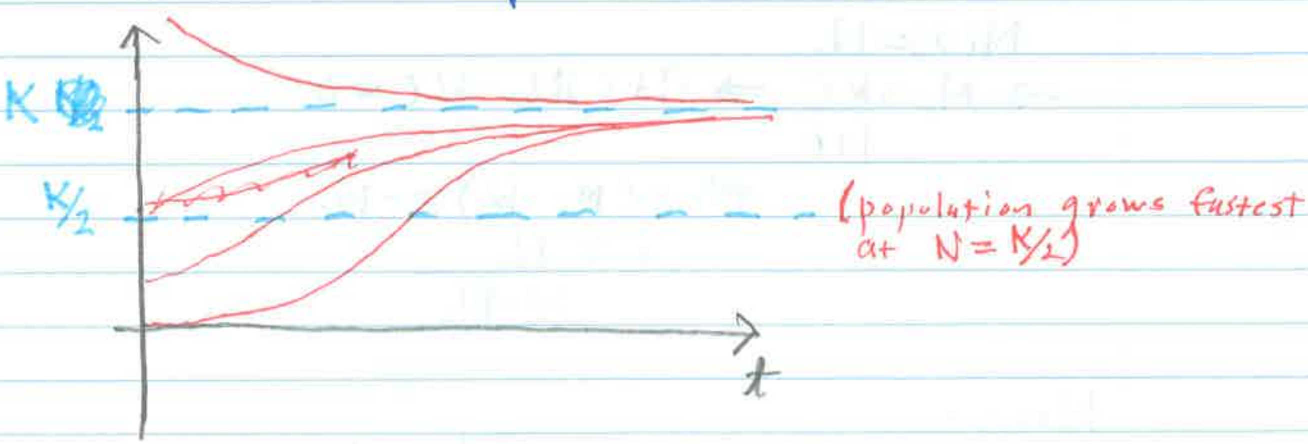
Model 4 (Logistic Growth):

$$N(t+\Delta t) = N(t) + r(K - N(t))N(t)\Delta t$$

$$\Rightarrow \frac{dN}{dt} = \lim_{\Delta t \to 0} \frac{N(t+\Delta t) - N(t)}{\Delta t} = r(K - N(t))N(t)$$

We will solve this equation momentarily. However, lets see what qualitative information we can learn from algebra.





Lets solve:

$$\int \frac{1}{(K-N)N}\, dN = \int r\, dt$$

$$\Rightarrow \int \left(\frac{1}{KN} + \frac{1}{K(K-N)} \right) dN = \int r\, dt$$

$$\frac{1}{K}\ln(N) - \frac{1}{K}\ln(K-N) + c = rt$$

$$\Rightarrow \frac{1}{K}\ln\left(\frac{N}{K-N}\right) + c = rt$$

$$\Rightarrow \ln\left(\frac{N}{K-N}\right) = rtK + c$$

$$\Rightarrow \frac{N}{K-N} = e^{rtK+c}$$

$$\Rightarrow \frac{N}{K-N} = Ce^{\bar{r}t} \qquad \bar{r} = rK, \; C = e^{c}$$

$$\Rightarrow N = (K-N)Ce^{\bar{r}t}$$

$$\Rightarrow N(1+Ce^{\bar{r}t}) = KCe^{\bar{r}t}$$

$$\Rightarrow N(t) = \frac{KCe^{\bar{r}t}}{1+Ce^{\bar{r}t}}, \; *$$

$N(0) = N_0$

$$\Rightarrow N_0 = \frac{KC}{1+C} \Rightarrow (1+C)N_0 - KC = 0$$

$$\Rightarrow C(N_0 - K) = -N_0$$

$$C = \frac{N_0}{K-N_0}$$

Now,

$$\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \frac{KCe^{\bar{r}t}}{1+Ce^{\bar{r}t}} = K$$

As $t \to \infty$ all intial populations go to the carraying capacity K.

Model 5 (Leslie Model)

For human populations it may be more useful to develop a model based on demographic information. That is, divide up the population by age.

Let

$N_0 = \#$ of people aged 0–5 years of age.

$N_i = \#$ of people aged $5i - 5(i+1)$ years of age

The population in t (5 year intervals) can be modeled by:

$$N_0(t+1) = \underbrace{N_1(t) b_1}_{\text{birth rate for ages 0-5 (hopefully 0)}} + N_2(t) b_2 + \underbrace{N_3(t) b_3}_{\text{birth rate ages 10-15}} + \ldots + \underbrace{N_{20}(t) b_{20}}_{\text{birth rate ages 95-100}}$$

$$N_1(t+1) = N_0(t) - \underbrace{d_0}_{\text{mortality rate ages 0-5}} N_0(t) = (1-d_0) N_0(t)$$

$$N_2(t+1) = N_1(t) - d_1 N_1(t) = (1-d_1) N_1(t)$$

$$\vdots$$

$$N_{20}(t+1) = (1-d_{20}) N_{19}(t).$$

This type of model is best analyzed using linear algebra.

One important question is how do b_i, d_i couple to substain the population.

Model 6 (Probability):

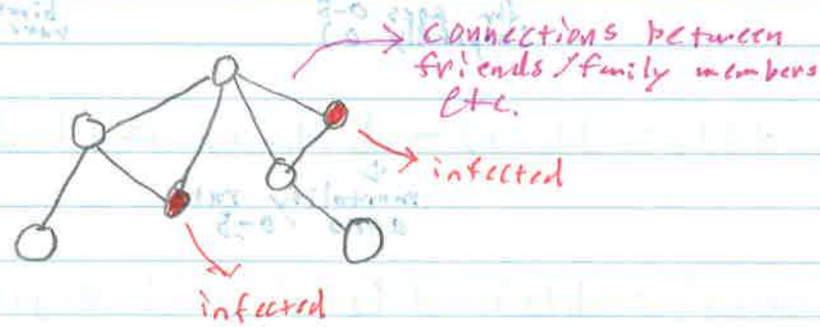
Another perhaps more realistic model is to model every individual. Label each individual by

N_i,

At each time step there is a probability p_b that the individual gives birth and a probability p_d that an individual dies.

Model 7 (Networks):

How can we model the spread of disease on a network



At each time step

1. probability you infect connections is p_I.
2. probability you recover p_R.

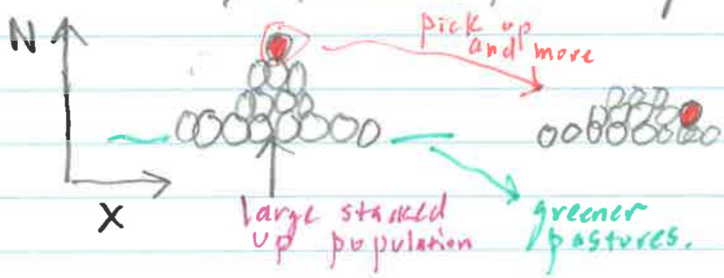
Model 8 (2-variables)

Lets adapt our population model to account for spatial localization of populations.

Old logistic model:

$$\frac{dN}{dt} = r(K-N)N$$

Relationship between rate of change in time and total population.
However densly packed populations grow differently!



Lets look at population at coordinate x:

$N(x,t)$

\# of people at coordinate x at time t.

$$\frac{dN}{dt} = r(K-N)N + D\frac{d^2N}{dX^2}$$

logistic growth

diffusion

→ population increases at points with high curvature.

→ population decreases at points with negative curvature.

Summary:

Building a Model:

Cycle between 1-4 as needed.

1. Formulate the problem. (What do you want to do?)
2. Outline the model. (Formulate possible variables and parameters)
3. Is it useful? (Can we obtain data to test the model)
4. Test the model. (Use the model to make predictions that can be tested against data/common sense.)