Math 150 Final Exam Spring 2005

[20] 1. Find the following limits. You must show all your work.

a) $\lim\limits_{x\to 3} \dfrac{x^2 - x - 6}{x^2 - 2x - 3}$ b) $\lim\limits_{x\to 0} \dfrac{3x}{\sin 2x}$

c) $\lim\limits_{x\to\infty} (x - \sqrt{x^2 - 3x + 4})$ d) $\lim\limits_{x\to 1} \dfrac{e^{x^2} - e^{1^2}}{x - 1}$

Answers

[10] 2. USING THE DEFINITION of derivative find the derivative of $f(x) = \dfrac{1}{x+2}$.

[30] 3. Find $f'(x)$ for the following functions. (You do not need to simplify your answers.)

a) $f(x) = x^2 + 2\tan^{-1} x + 3\tan 5x$ b) $f(x) = (\sinh x)(1 + e^x)$

c) $f(x) = \dfrac{\sin x - 1}{5 + 3e^x}$ d) $f(x) = (3x^3 + 5x - 4)^{3/5}$

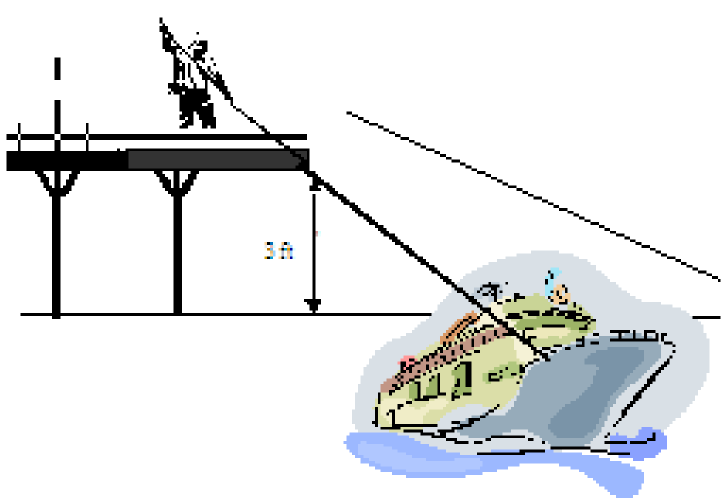
Find $f'(x)$.

e) $f(x) = x^{\ln x}$ f) $f(x) = \displaystyle\int_1^x (t^3 + 1)^{1/2} dt$

[8] 4. Let $y(x)$ be defined implicitly by $x^2 + y^3 = e^y$. Find $y'(x)$.

[12] 5. Determine an equation of the tangent line to the curve $f(x) = e^{x^2+x}$ at the point where the graph crosses the y-axis.

[10] 6. A boat is pulled into a dock by means of a winch 3 feet above the deck of the boat. The winch pulls in rope at a rate of 4 feet per second. Determine the speed of the boat when there is 5 feet of rope out.



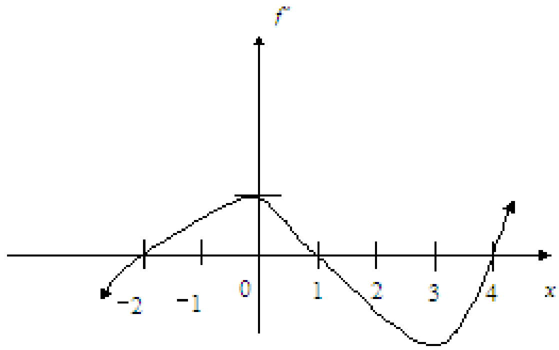
Answers

[10] 7. Find the absolute maximum value of $f(x) = \ln x + \frac{1}{x}$ on $\left[\frac{1}{e}, e\right]$.

Answers

[10] 8. A page is to contain 24 square inches of printed area. The margins at left, right and bottom are 1 inch. The margin at the top is 2 inches. Find the dimensions of the page such that the least amount of paper is used.

[15] 9. The graph of the DERIVATIVE, f', of a function f is given by



a) Find the open intervals where f is increasing and those where f is decreasing.

b) Find the open intervals where f is concave up and those where f is concave down.

c) Suppose $f(0) = 1$. Graph a plausible f using the above information.

Answers

[25] 10. Evaluate the following indefinite integrals.

a) $\int (x^2 - 4\sqrt{x} + 3)dx$ b) $\int \left(\frac{x^2 - 5x + 4}{x}\right) dx$ c) $\int \sec^2(5x) - \cos(5x)\, dx$

d) $\int \frac{x}{\sqrt{1 - x^4}}\, dx$ e) $\int \frac{e^{2x}}{(e^{2x}+1)^3}\, dx$

[18] 11. Evaluate the following definite integrals.

a) $\int_0^{\pi/2} e^{\cos x} \sin x dx$ b) $\int_0^{\pi/4} \frac{\sec x \tan x}{1 + \sec x} dx$ c) $\int_0^3 |x - 1| dx$

Answers

[10] 12. Find the area of the region bounded by the curves $f(x) = x^3 - 3x$ and $g(x) = x$.

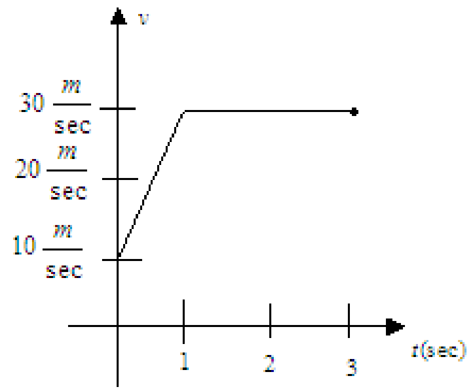
Answers

[14] 13. Let R be the region bounded by the curves $y = \cos x$, $y = 0$, $x = 0$, and $x = \frac{\pi}{4}$.

a) Set up, but DO NOT EVALUATE, an integral for the volume of the solid obtained by revolving this region around the $y-$axis.

b) Set up, but DO NOT EVALUATE, an integral for the volume of the solid obtained by revolving this region around the line $y = -1$.

[8] 14. The velocity function (in meters per second) for a particle moving along a line is given by



Answers

a) Find the acceleration at $t = \frac{1}{2}$ sec.

b) Find the TOTAL DISTANCE traveled on the time interval $0 \leq t \leq 3$ sec.

1a) $\lim_{x\to3}\frac{x^2-x-6}{x^2-2x-3}=\frac{3^2-3-6}{3^2-2\cdot3-3}=\frac{0}{0}$

$=\lim_{x\to3}\frac{(x-3)(x+2)}{(x-3)(x+1)}=\lim_{x\to3}\frac{x+2}{x+1}=\frac{5}{4}$

b) $\lim_{x\to0}\frac{3x}{\sin 2x}=3\lim_{x\to0}\frac{2x}{\sin 2x}\cdot\frac{1}{2}$

$=\frac{3}{2}\lim_{x\to0}\frac{2x}{\sin 2x}=\frac{3}{2}\cdot1$

c) $\lim_{x\to\infty}(x-\sqrt{x^2-3x+4})$

$=\lim_{x\to\infty}(x-\sqrt{x^2-3x+4})\left[\frac{x+\sqrt{x^2-3x+4}}{x+\sqrt{x^2-3x+4}}\right]$

$=\lim_{x\to\infty}\frac{x^2-(x^2-3x+4)}{x+\sqrt{x^2-3x+4}}=\lim_{x\to\infty}\frac{3x-4}{x+\sqrt{x^2-3x+4}}$

$=\lim_{x\to\infty}\frac{x\left(3-\frac{4}{x}\right)}{\left(1+\sqrt{1-\frac{3}{x}+\frac{4}{x^2}}\right)}=\frac{3}{1+1}=\frac{3}{2}$

d) $\lim_{x\to1}\frac{e^{x^2}-e^{1^2}}{x-1}=\frac{e^{1^2}-e^{1^2}}{1-1}=\frac{0}{0}$

$\lim_{x\to1}\frac{(e^{x^2}-e^{1^2})}{(x-1)}\frac{(x+1)}{(x+1)}$

$=\lim_{x\to1}(x+1)\lim_{x\to1}\frac{e^{x^2}-e^{1^2}}{x^2-1}$

$=(1+1)\cdot1=2$

Return to Pg 1

2. $f'=\lim_{h\to0}\frac{\frac{1}{x+h+2}-\frac{1}{x+2}}{h}=\lim_{h\to0}\frac{\frac{x+2-(x+h+2)}{(x+h+2)(x+2)}}{h}$

$=\lim_{h\to0}\frac{-h}{(x+h+2)(x+2)h}=\frac{-1}{(x+2)^2}$

3. a) $f'=2x+\frac{2}{1+x^2}+3(\sec^2 5x)\cdot5$

b) $f'=(\cosh x)(1+e^x)+\sinh x(e^x)$

c) $f'=\frac{\cos x(5+3e^x)-(\sin x-1)3e^x}{(5+3e^x)^2}$

d) $f'=\frac{3}{5}(3x^3+5x-4)^{-2/5}(9x^2+5)$

e) $y=x^{\ln x}$

$\ln y=(\ln x)(\ln x)=(\ln x)^2$

$\frac{1}{y}y'=\frac{2\ln x}{x}\Rightarrow f'=x^{\ln x}\left(\frac{2\ln x}{x}\right)$

f) $f'=(x^3+1)^{1/2}$

4. $2x+3y^2y'=e^y\cdot y'$

$\Rightarrow 2x=y'(e^y-3y^2)$

$\Rightarrow y'=\frac{2x}{e^y-3y^2}$

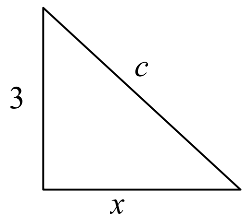
Return to Pg 1

5. When f crosses the y-axis, $x = 0$ and $f(0) = 1$

$$f' = e^{x^2+x}(2x+1)$$
$$f'(0) = e^0(2\cdot 0+1) = 1$$
$$y - 1 = 1(x-0)$$

Return to Pg 1

6.



$$c^2 = x^2 + 3^2$$
$$2c\frac{dc}{dt} = 2x\frac{dx}{dt}$$

$$\frac{dc}{dt} = -4$$

when $c = 5 \Rightarrow x = 4$

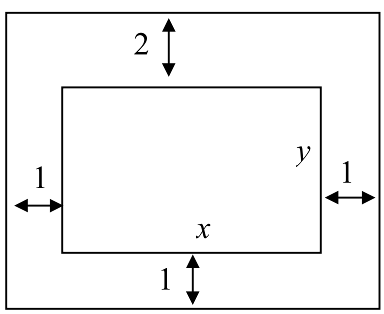
$$\frac{dx}{dt} = \frac{c}{x}(-4)$$
$$\frac{dx}{dt} = \frac{5}{4}(-4) = -5$$

7. $f' = \frac{1}{x} - \frac{1}{x^2} = 0$

$$\frac{x-1}{x^2} = 0 \Rightarrow x = 1$$

$$f\left(\frac{1}{e}\right) = -1 + e \quad \text{(maximum)}$$
$$f(1) = 1$$
$$f(e) = 1 + \frac{1}{e}$$

8.



$xy = 24$ $\qquad A = (x+2)(y+3)$

$\Rightarrow y = \frac{24}{x}$ $\qquad A = (x+2)\left(\frac{24}{x}+3\right) = 24 + 3x + \frac{48}{x} + 6$

$$A' = \frac{-48}{x^2} + 3 = 0 \Rightarrow \frac{48}{x^2} = 3 \Rightarrow x^2 = 16,\ x = \pm 4$$

$A'' = \frac{96}{x^3}$ $\qquad A''(4) > 0 \Rightarrow x = 4$ is a min

$x = 4 \quad y = 6$

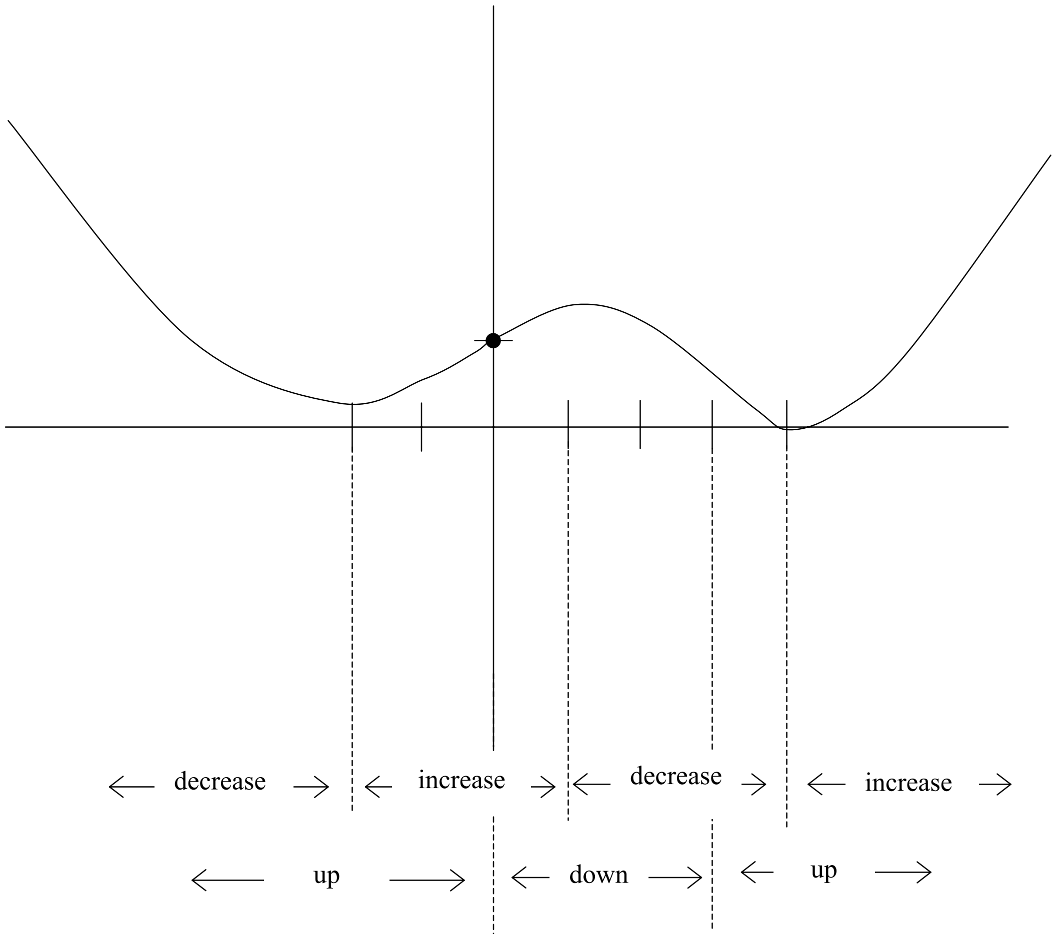
Return to Pg 2

9a) f is increasing on $(-2,1)\cup(4,\infty)$
f is decreasing on $(-\infty,-2)\cup(1,4)$

b) f is concave up on $(-\infty,0)\cup(3,\infty)$
f is concave down on $(0,3)$

c)

Return to Pg 2



10a) $\int (x^2 - 4\sqrt{x} + 3)dx = \int (x^2 - 4x^{1/2} + 3)dx = \frac{x^3}{3} - \frac{4x^{3/2}}{3/2} + 3x + C = \frac{x^3}{3} - \frac{8}{3}x^{3/2} + 3x + C$

b) $\int \left(\frac{x^2 - 5x + 4}{x} \right) dx = \int \left(x - 5 + \frac{4}{x} \right) dx = \frac{x^2}{2} - 5x + 4\ln|x| + C$

c) Let $u = 5x$, $du = 5dx$

$\int (\sec^2 5x - \cos 5x)dx = \int (\sec^2 u - \cos u)\frac{du}{5} = \frac{1}{5}[\tan u - \sin u] + C = \frac{1}{5}[\tan 5x - \sin 5x] + C$

Return to Pg 2

10d) $u = x^2, \quad du = 2x\,dx$

$$\int \frac{x\,dx}{\sqrt{1-x^2}} = \int \frac{\frac{1}{2}du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}u + C = \frac{1}{2}\sin^{-1}x^2 + C$$

Return to Pg 2

e) $u = e^{2x} + 1, \quad du = 2e^{2x}dx$

$$\int \frac{e^{2x}dx}{(e^{2x}+1)^3} = \int \frac{\frac{1}{2}du}{u^3} = \frac{1}{2}\int u^{-3}du = \frac{1}{2}\frac{u^{-2}}{-2} + C = \frac{-1}{4(e^{2x}+1)^2} + C$$

11a) $u = \cos x$ when $x = 0 \Rightarrow u = 1$, when $x = \frac{\pi}{2} \Rightarrow u = 0$

$du = -\sin x\,dx$

$$\int_0^{\pi/2} e^{\cos x}\sin x\,dx = \int_1^0 e^u(-du) = \int -e^u \Big|_{u=1}^{0} = -e^0 - (-e^1) = e - 1$$

b) $u = 1 + \sec x$ when $x = 0 \Rightarrow u = 2$, $x = \frac{\pi}{4} \Rightarrow u = 1 + \sqrt{2}$

$du = \sec x \tan x\,dx$

$$\int_0^{\pi/4} \frac{\sec x \tan x}{1+\sec x}dx = \int_2^{1+\sqrt{2}} \frac{du}{u} = \ln u \Big|_{u=2}^{1+\sqrt{2}} = \ln(1+\sqrt{2}) - \ln 2$$

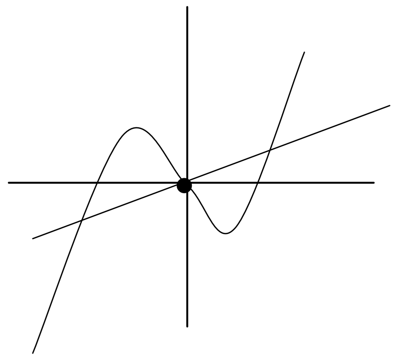
c) $\displaystyle\int_0^3 |x-1|\,dx = \int_0^1 -(x-1)dx + \int_1^3 (x-1)dx$

$$= \left(-\frac{x^2}{2} + x\right)\Bigg|_{x=0}^{1} + \left(\frac{x^2}{2} - x\right)\Bigg|_{x=1}^{3} = -\frac{1}{2} + 1 - (0) + \left(\frac{3^2}{2} - 3\right) - \left(\frac{1^2}{2} - 1\right)$$

$$= \frac{1}{2} + \frac{9}{2} - 3 + \frac{1}{2} = \frac{3}{2}$$

Return to Pg 2

12.



$$x^3 - 3x = x \Rightarrow x^3 - 4x = 0 \Rightarrow x = 0_1 \pm 2$$

Return to Pg 2

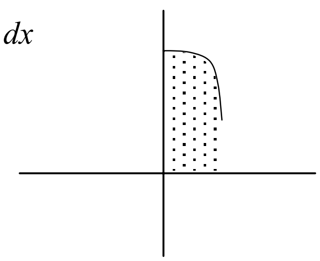
$$\int_{-2}^{0} (x^3 - 3x - x)dx + \int_{0}^{2} (x - (x^3 - 3x))dx$$

$$= \left(\frac{x^4}{4} - \frac{4x^2}{2}\right)\Bigg|_{x=-2}^{0} + \left(\frac{4x^2}{2} - \frac{x^4}{4}\right)\Bigg|_{x=0}^{2}$$

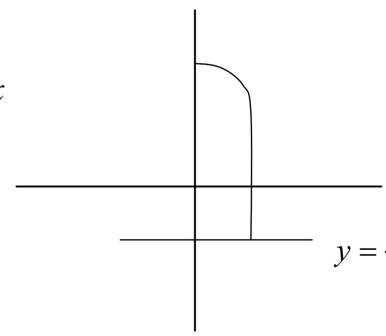
$$= 0 - \left(\frac{(-2)^4}{4} - \frac{4(-2)^2}{2}\right) + \frac{4 \cdot 2^2}{2} - \frac{2^4}{4} - 0$$

$$= -(4-8) + 8 - 4 = 8$$

13a) $\displaystyle\int_{0}^{\pi/4} 2\pi\, x \cos x \, dx$



b) $\displaystyle\int_{0}^{\pi/4} \pi(\cos x + 1)^2 dx$



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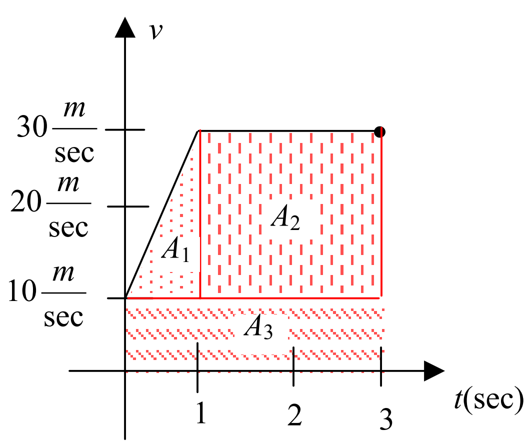

a) $\dfrac{v(1) - v(0)}{1 - 0} = \dfrac{30 - 10}{1} = 20 \dfrac{\text{m}}{\text{sec}^2}$

b)

Area $A_1 = \frac{1}{2} \cdot 20 \cdot 1 = 10$

Area $A_2 = 20 \cdot 2 = 40$

Area $A_3 = 10 \cdot 3 = 30$

$$A_1 + A_2 + A_3 = 80$$

Return to Pg 3