First Passage Times of a Jump Diffusion Process

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Abstract

This paper studies the first passage times to flat boundaries for a double exponential jump diffusion process, which consists of a continuous part driven by a Brownian motion and a jump part with jump sizes having a double exponential distribution. Explicit solutions of the Laplace transforms, of both the distribution of the first passage times and the joint distribution of the process and its running maxima, are obtained. Because of the overshoot problems associated with general jump diffusion processes, the double exponential jump diffusion process offers a rare case in which analytical solutions for the first passage times are feasible. In addition, it leads to several interesting probabilistic results. Numerical examples are also given. The finance applications include pricing barrier and lookback options.

Key words: Renewal theory, martingale, differential equations, integral equations, infinitesimal generators, marked point processes, Lévy processes, Gaver-Stehfest algorithm.
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1 Introduction

Jump diffusion processes are processes of the form

$$X_t = \sigma W_t + \sum_{i=1}^{N_t} Y_i + \mu t, \qquad X_0 \equiv 0.$$

Here $\{W_t;\ t \geq 0\}$ is a standard Brownian motion with $W_0 = 0$, $\{N_t;\ t \geq 0\}$ is a Poisson process, constants μ and $\sigma > 0$ are the drift and volatility of the diffusion part, respectively, and the jump sizes $Y_1, Y_2, \cdots$, are independent identically distributed random variables. They are used, for example, in finance to model asset (stock, bond, currency, etc.) prices. Two examples are the normal jump diffusion process (Merton, 1976) where Y has a normal distribution, and the double exponential jump diffusion process (Kou, 1999) where Y has a double exponential distribution (could be asymmetric).

This paper focuses on the first passage times for the double exponential jump diffusion process:

$$\tau_b := \inf\{t \geq 0; \quad X_t \geq b\}, \quad b > 0.$$

The main problems studied include the distribution of the first passage time

$$\mathsf{P}(\tau_b \leq t), \tag{1.1}$$

for all $t > 0$, the joint distribution between the first passage time and the terminal value

$$\mathsf{P}(\tau_b \leq t, \ X_t \geq a), \tag{1.2}$$

and the joint distributions between the overshoot and the first passage time

$$\mathsf{P}(\tau_b \leq t, \ X_{\tau_b} - b > y), \quad y \geq 0, \qquad \mathsf{P}(\tau_b \leq t, \ X_{\tau_b} - b = 0), \tag{1.3}$$

As a by product, we also compute $\mathsf{P}(\tau_b < \infty)$, $\mathsf{E}(\tau_b 1_{\{\tau_b < \infty\}})$, $\mathsf{P}(\tau_b < \infty, \ X_{\tau_b} - b = 0)$, and $\mathsf{P}(\tau_b < \infty, \ X_{\tau_b} - b > y)$ for $y \geq 0$, explicitly.

There are three reasons why these problems are interesting. First, from a purely probabilistic point of view, the double exponential jump diffusion process offers a rare case in which analytical solutions of the first passage times are feasible. Because of the jump part, when a jump diffusion process crosses the boundary level b, sometimes it hits the boundary exactly and sometimes it may incur an "overshoot" over the boundary. In general, the distribution of the overshoot and the dependent structures between the overshoot and the original jump diffusion process are not known analytically, thus making it impossible to get closed form solutions of the distribution of the first passage times. However, if the jump sizes, Y's, have an exponential-type distribution, the overshoot problems can be solved analytically, thanks to the special feature of the memoryless property associated with the exponential distribution. See Siegmund (1985, Ch. 8) and Woodroofe (1982) for some detailed discussions of overshoot problems.

Second, the study leads to several interesting probabilistic results. (1) Although the exponential random variables have memoryless properties, the first passages time and the overshoot are dependent, despite the fact that the two are conditionally independent given that the overshoot is bigger than 0. (2) The renewal-type integral equations, which are used frequently in studying first passage times, may not lead to unique solutions for the problems, because the boundary conditions are difficult to determine; see Section 4.3. Instead, our approach based on differential equations and martingales can circumvent this problem of uniqueness.

Third, from the applied probability point of view, the results of this paper are useful in option pricing. Brownian motion and normal distribution have been widely used, for example, in Black-Scholes option pricing framework, to study the return of assets. However, two puzzles, that emerged from many empirical investigations, have recently received a great deal of attention, namely the leptokurtic feature, meaning that the return distribution of assets may have a higher peak and two (asymmetric) heavier tails than those of the normal distribution, and an empirical abnormality called "volatility smile" in option pricing. Many studies have been conducted to modify the Black-Scholes models, in order to explain the two puzzles. To incorporate the leptokurtic and asymmetric features, a variety of models have been proposed, including, among others, (a) chaos theory, fractal Brownian motion, and stable processes; (b) generalized hyperbolic models, including log t model and log hyperbolic model; (c) time changed Brownian motions. In a parallel development, different models are also proposed to incorporate the "volatility smile." Popular ones are (a) stochastic volatility and ARCH models; (b) constant elasticity model (CEV model); (c) normal jump diffusion model; (d) implied binomial trees. See the textbooks by Hull (2000) and Duffie (1998) for more details.

An immediate problem with these models is that it may be difficult to obtain analytical solutions for the purpose of option pricing; more precisely, they might give some analytical formulae for the regular call and put options, but certainly not for the popular path-dependent options, such as perpetual American options, barrier and lookback options. To get analytically tractable models, and to incorporate both the leptokurtic feature and the "volatility smile," the double exponential jump diffusion model is proposed by Kou (1999); see also Glasserman and Kou (1999) for pricing of interest rate derivatives under such model and more background about general jump diffusion models.

The explicit calculation of (1.1) and (1.2) for the first passage time can be used to get closed form solutions for pricing barrier and lookback options under the double exponential jump diffusion model. The details of its finance applications, being too long to be included here, will be reported in Kou and Wang (2001).

In this paper, we shall demonstrate that it is possible to compute the Laplace transform for (1.1), (1.2), and (1.3) explicitly. In the case of (1.2), the Laplace transform is given in terms of a special function called Hh function, which can be computed easily via a linear recursion. Using a Laplace inverse algorithm (the Gaver-Stehfest algorithm), both (1.1) and (1.2) can then be computed very fast; see Section 6 for details of Laplace inversion and some numerical examples.

An outline of the paper is as follows. In Section 2 the double exponential jump diffusion process is introduced, and intuition about the closed form solutions is also given. Section 3 gives some preliminary results. Section 4 presents the computation of the Laplace transform of the first passage times, its immediate corollaries, and its connection with the integral equation approach. The joint distribution of the jump diffusion process and its running maxima is considered in Section 5, which also includes a brief account of Hh functions. Inversion of Laplace transforms and numerical examples are given in Section 6. Some proofs and technical details are deferred to Appendices to ease exposition.

2 Background and Intuition

Consider a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t; t \geq 0\}, \mathsf{P})$ where the filtration satisfies the usual conditions. The double exponential jump diffusion process analyzed in this paper consists of two parts, a continuous part driven by a Brownian motion, and a jump part with jump sizes having a common double exponential distribution. More precisely,

$$X_t = \sigma W_t + \mu t + \sum_{i=1}^{N_t} Y_i; \qquad X_0 \equiv 0. \tag{2.1}$$

Here $\{W_t;\ t \geq 0\}$ is a standard Brownian motion, and $\{N_t;\ t \geq 0\}$ is a Poisson process with intensity λ. Constants μ and $\sigma > 0$ represent the drift and volatility of the continuous part, respectively. The jump sizes $\{Y_1, Y_2, \cdots\}$ form a sequence of independent identically distributed (i.i.d.) random variables with the common density

$$f_Y(y) = p \cdot \eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q \cdot \eta_2 e^{\eta_2 y} 1_{\{y < 0\}}, \tag{2.2}$$

where constants $p, q \geq 0$, $p + q = 1$ and $\eta_1, \eta_2 > 0$. Note that the means of the two exponential distributions are $1/\eta_1$ and $1/\eta_2$. We also assume that the random processes $\{W_t;\ t \geq 0\}$, $\{N_t;\ t \geq 0\}$ and random variables $\{Y_1, Y_2, \cdots\}$ are independent. The jump part of this process, $\sum_{i=1}^{N_t} Y_i$, is a special case of the so-called marked point processes; further background on marked point processes can be found, for example, in Brémaud (1981) and Jacod and Shiryaev (1987).

We are interested in analyzing the first passage time to level b, defined by

$$\tau_b \stackrel{\triangle}{=} \inf\{t \geq 0; \quad X_t \geq b\}, \quad b > 0, \tag{2.3}$$

where

$$X_{\tau_b} \stackrel{\triangle}{=} \limsup_{t \to \infty} X_t, \quad \text{on the set } \{\tau_b = \infty\}.$$

More precisely, we shall examine the distribution of the first passage time

$$\mathsf{P}(\tau_b \leq t) = \mathsf{P}(\max_{0 \leq s \leq t} X_s \geq b) \tag{2.4}$$

for all $t \geq 0$, and the joint distribution of the first passage time and the terminal value,

$$\mathsf{P}(\tau_b \leq t,\ X_t \geq a),$$

and their related quantities.

Without the jump part, the process simply becomes a Brownian motion with drift μ. The distribution of the first passage times and the joint distribution between the first passage times and the terminal values for a Brownian motion with drift are well known. They can be obtained either by a combination of a change of measure (Girsanov theorem) and the reflection principle, or by calculating the Laplace transforms via some appropriate martingales and optional sampling theorem. Details of both methods can be found in many classical textbooks on stochastic analysis; see, for example, Karatzas and Shreve (1991), and Karlin and Taylor (1975).

With the jump part, however, it is very difficult to study the first passage times for general jump diffusion processes. When a jump diffusion process crosses boundary level b, sometimes it hits the boundary exactly and sometimes it incurs an "overshoot", $X_{\tau_b} - b$, over the boundary. See Fig. 1 for an illustration.

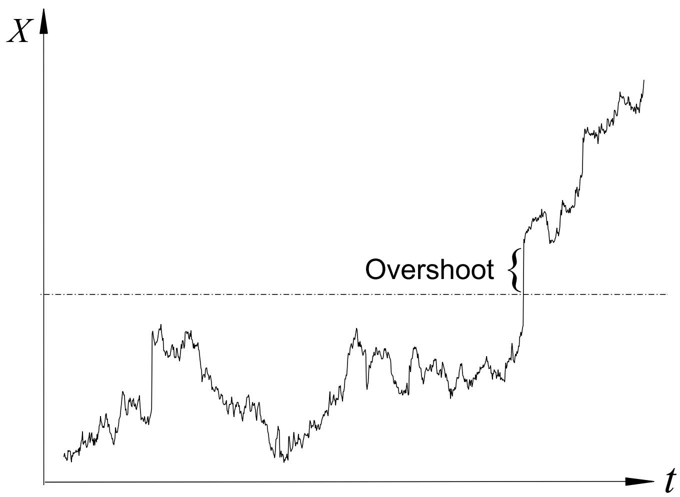

Figure 1: A Simulated Sample Path with the Overshoot Problem

The overshoot presents several problems, if one wants to compute the distribution of the first passage times analytically.

- One needs to get the exact distribution of the overshoot, $X_{\tau_b} - b$; particularly, $\mathsf{P}(X_{\tau_b} - b = 0)$ and $\mathsf{P}(X_{\tau_b} - b > x)$, $x > 0$. This is only possible if the jump size Y has an exponential type distribution, thanks to the memoryless property of the exponential distribution.

- One needs to know the dependent structure between the overshoot, $X_{\tau_b} - b$, and the first passage time τ_b. The two random variables are conditionally independent, given that the overshoot is bigger than 0, if the jump size Y has an exponential type distribution, again thanks to the memoryless property. This conditionally independent structure seems to be very special to the exponential distribution, and does not hold for general distributions.

- If one wants to use the reflection principle to study the first passage times, the dependent structure between the overshoot and the terminal value X_t is also needed afterwards. This is not known to the best of our knowledge, even for the double exponential jump diffusion process.

Consequently, we can derive closed form solutions for the Laplace transforms of the first passage times for the double exponential jump diffusion process, yet cannot give more explicit calculations beyond that, as the correlation between X_t and $X_{\tau_b} - b$ is not available. However, for other jump diffusion processes, even analytical forms of the Laplace transforms seem to be quite difficult, if not impossible.

To compute the Laplace transform of $\mathsf{P}(\tau_b \leq t)$, we use both martingale and differential equations. There are two other possible approaches: renewal-type integral equations and Wiener-Hopf factorization. Renewal-type integral equations are frequently used in actuarial science literature (see, for example, Gerber and Landry, 1998, and reference therein) to study first passage times. However, (1) most of actuarial literature is devoted to one-sided jump distributions (thus it might not have the overshoot problems, if, for example, the jump size can only be negative and the barrier is positive), and our double exponential distribution is two-sided (can jump both up and down); (2) the renewal equations may not lead to unique solutions (see Section 4.3 for details), and thus would not solve the problems in our case.

Wiener-Hopf factorization has been widely used to study the first passage times for Lévy processes (note that the double exponential jump diffusion process is a special case of the Lévy processes); for a survey, see, for example, Sato (1999). Many such studies focus on one-sided jumps; see, for example, Rogers (2000). However, because of the one-sided jumps, the “overshoot” problems are avoided, as the jumps are in the opposite direction of the barrier crossing. A closely related paper for two-sided jumps is Boyarchenko and Levendorski (2000), in which they discuss the Wiener-Hopf factorization for general jump diffusion processes. In general, explicit calculation of the Wiener-Hopf factorization is difficult. Because of the special structure of the exponential distribution, especially due to its memoryless property, we can solve the first passage time problems explicitly. In some sense, that also suggests, though indirectly, that the Wiener-Hopf factorization could be performed explicitly in the case of double exponential jump diffusion processes.

After the Laplace transform of τ_b is calculated, the Laplace transform of $\mathsf{P}(\tau_b \le t,\ X_t \ge a)$ can be obtained as a product of the Laplace transforms of τ_b and X_t. Most of Section 5 is devoted to computing the Laplace transform of X_t by using the Hh function.

3 Preliminary Results

The infinitesimal generator of the jump diffusion process (2.1) is given by

$$\mathcal{L}u(x) = \frac{1}{2}\sigma^2 u''(x) + \mu u'(x) + \lambda \int_{-\infty}^{\infty} [u(x+y) - u(x)]\, f_Y(y)\, dy, \tag{3.1}$$

for all twice continuously differentiable functions $u(x)$. In addition, suppose $\theta \in (-\eta_2, \eta_1)$. The moment generating function of jump size Y is given by

$$\mathsf{E}\left[e^{\theta Y}\right] = \frac{p\eta_1}{\eta_1 - \theta} + \frac{q\eta_2}{\eta_2 + \theta},$$

from which the moment generating function of X_t can be obtained as

$$\phi(\theta, t) \triangleq \mathsf{E}\left[e^{\theta X_t}\right] = \exp\{G(\theta)t\}, \tag{3.2}$$

where the function $G(\cdot)$ is defined as

$$G(x) \triangleq x\mu + \frac{1}{2}x^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - x} + \frac{q\eta_2}{\eta_2 + x} - 1\right). \tag{3.3}$$

Lemma 3.1. *The equation*

$$G(x) = \alpha, \ \ \forall \alpha > 0,$$

has exactly four roots: $\beta_{1,\alpha}$, $\beta_{2,\alpha}$, $-\beta_{3,\alpha}$, $-\beta_{4,\alpha}$, *where*

$$0 < \beta_{1,\alpha} < \eta_1 < \beta_{2,\alpha} < \infty, \ \ 0 < \beta_{3,\alpha} < \eta_2 < \beta_{4,\alpha} < \infty.$$

In addition, let the overall drift of the jump diffusion process be

$$\bar{u} \triangleq \mu + \lambda\left(\frac{p}{\eta_1} - \frac{q}{\eta_2}\right).$$

Then as $\alpha \to 0$,

$$\beta_{1,\alpha} \to \begin{cases} 0, & \text{if } \bar{u} \ge 0 \\ \beta_1^*, & \text{if } \bar{u} < 0 \end{cases}, \qquad \beta_{2,\alpha} \to \beta_2^*,$$

where β_1^* *and* β_2^* *are defined as the unique roots*

$$G(\beta_1^*) = 0, \ \ G(\beta_2^*) = 0, \ \ 0 < \beta_1^* < \eta_1 < \beta_2^* < \infty.$$

Proof. Since $G(\beta)$ is a convex function on interval $(-\eta_2, \eta_1)$ with $G(0) = \lambda \cdot (p+q-1) = 0$, and $G(\eta_1-) = +\infty$, $G(-\eta_2+) = +\infty$, there is exactly one root $\beta_{1,\alpha}$ for $G(x) = \alpha$ on the interval $(0,\ \eta_1)$, and another one on the interval $(-\eta_2,\ 0)$. Furthermore, Since $G(\eta_1+) = -\infty$, $G(+\infty) = \infty$, there is at least one root on (η_1, ∞). Similarly, there is at least one root on $(-\infty, -\eta_2)$, as $G(-\infty) = \infty$ and $G(-\eta_2-) = -\infty$. But the equations $G(\beta) = \alpha$ is actually a polynomial equation with degree four; therefore, the equation can have at most four real roots. It follows that on each interval, $(-\infty, -\eta_2)$ and (η_1, ∞), there is exactly one root.

The limiting results when $\alpha \to 0$ follow easily, once we note that $G'(0) = \bar{u}$. □

The following result shows that the memoryless property of the random walk of exponential random variables leads to the conditional memoryless property of the jump diffusion process.

Proposition 3.1. *(Conditional Memoryless Property). For any $x > 0$,*

$$\mathsf{P}(\tau_b \le t,\ X_{\tau_b} - b \ge x) = e^{-\eta_1 x}\mathsf{P}(\tau_b \le t,\ X_{\tau_b} - b > 0), \tag{3.4}$$

$$\mathsf{P}(X_{\tau_b} - b \ge x | X_{\tau_b} - b > 0) = e^{-\eta_1 x}, \tag{3.5}$$

Furthermore, conditional on $X_{\tau_b} - b > 0$, the stopping time τ_b and the overshoot $X_{\tau_b} - b$ are independent; more precisely, for any $x > 0$,

$$\mathsf{P}(\tau_b \le t,\ X_{\tau_b} - b \ge x | X_{\tau_b} - b > 0) = \mathsf{P}(\tau_b \le t | X_{\tau_b} - b > 0)\mathsf{P}(X_{\tau_b} - b \ge x | X_{\tau_b} - b > 0), \tag{3.6}$$

Proof. We only need to show that equality (3.4) holds. Equality (3.5) follows readily by letting $t \to \infty$ and observe that on set $\{X_{\tau_b} > b\}$, the hitting time τ_b is finite by definition; and equality (3.6) also holds since

$$\begin{aligned}
\mathsf{P}(\tau_b \le t,\ X_{\tau_b} - b \ge x | X_{\tau_b} - b > 0) &= \frac{\mathsf{P}(\tau_b \le t,\ X_{\tau_b} - b \ge x)}{\mathsf{P}(X_{\tau_b} - b > 0)} = e^{-\eta_1 x}\frac{\mathsf{P}(\tau_b \le t,\ X_{\tau_b} - b > 0)}{\mathsf{P}(X_{\tau_b} - b > 0)} \\
&= \mathsf{P}(X_{\tau_b} - b \ge x | X_{\tau_b} - b > 0)\mathsf{P}(\tau_b \le t | X_{\tau_b} - b > 0).
\end{aligned}$$

Deonte by $T_1,\ T_2, \ldots,$ the arrival times of the Poisson process N. It follows that

$$\mathsf{P}(\tau_b \le t,\ X_{\tau_b} - b \ge x) = \sum_{n=1}^{\infty} \mathsf{P}(T_n = \tau_b \le t,\ X_{T_n} - b \ge x) := \sum_{n=1}^{\infty} P_n,$$

as the overshoot insider the probability can only occur during the arrive times of the Poisson process, because $x > 0$. However, with $X_0(t) = \sigma W_t + \mu t$, we have

$$
\begin{aligned}
P_n &= \mathsf{P}(\max_{0\le s<T_n} X_s < b,\ X_{T_n} \ge b+x,\ T_n \le t) \\
&= \mathsf{E}\{\mathsf{P}(X_{T_n} \ge b+x|\mathcal{F}_{T_n-}, T_n)1_{\{\max_{0\le s<T_n} X_s<b,\, T_n\le t\}}\} \\
&= \mathsf{E}\{p\exp\{-\eta_1(b+x-X_0(T_n)-\xi_1-\cdots-\xi_{n-1})\}1_{\{\max_{0\le s<T_n} X_s<b,\, T_n\le t\}}\} \\
&= e^{-\eta_1 x}\mathsf{E}\{p\exp\{-\eta_1(b-X_0(T_n)-\xi_1-\cdots-\xi_{n-1})\}1_{\{\max_{0\le s<T_n} X_s<b,\, T_n\le t\}}\} \\
&= e^{-\eta_1 x}\mathsf{E}\{\mathsf{P}(X_{T_n} > b|\mathcal{F}_{T_n-}, T_n)1_{\{\max_{0\le s<T_n} X_s<b,\, T_n\le t\}}\} \\
&= e^{-\eta_1 x}\mathsf{P}(\max_{0\le s<T_n} X_s < b,\ X_{T_n} > b,\ T_n \le t) \\
&= e^{-\eta_1 x}\mathsf{P}(X_{T_n} - b > 0,\ T_n = \tau_b \le t).
\end{aligned}
$$

It follows that

$$
\mathsf{P}(\tau_b \le t,\ X_{\tau_b} - b \ge x) = \sum_{n=1}^{\infty} e^{-\eta_1 x}\mathsf{P}(T_n = \tau_b \le t,\ X_{\tau_b} - b > 0) = e^{-\eta_1 x}\mathsf{P}(\tau_b \le t,\ X_{\tau_b} - b > 0).
$$

This completes the proof. □

It is easy to verify from the conditional memoryless property that, for any $x > 0$, the following equalities hold:

$$
\begin{aligned}
\mathsf{P}(X_{\tau_b} - b \ge x) &= e^{-\eta_1 x}\mathsf{P}(X_{\tau_b} - b > 0), \\
\mathsf{E}(e^{-\alpha\tau_b}1_{\{X_{\tau_b} \ge b+x\}}) &= e^{-\eta_1 x}\mathsf{E}(e^{-\alpha\tau_b}1_{\{X_{\tau_b} - b > 0\}}).
\end{aligned}
$$

4 Distribution of the First Passage Times

4.1 The Laplace Transforms

Theorem 4.1. *For any $\alpha \in (0,\infty)$, let $\beta_{1,\alpha}$ and $\beta_{2,\alpha}$ be the only two positive roots for the equation*

$$
\alpha = G(\beta),
$$

where $0 < \beta_{1,\alpha} < \eta_1 < \beta_{2,\alpha} < \infty$. Then we have the following results regarding the Laplace transforms of τ_b and X_{τ_b}:

$$
\mathsf{E}[e^{-\alpha\tau_b}] = \frac{\eta_1 - \beta_{1,\alpha}}{\eta_1} \cdot \frac{\beta_{2,\alpha}}{\beta_{2,\alpha} - \beta_{1,\alpha}} e^{-b\beta_{1,\alpha}} + \frac{\beta_{2,\alpha} - \eta_1}{\eta_1} \cdot \frac{\beta_{1,\alpha}}{\beta_{2,\alpha} - \beta_{1,\alpha}} e^{-b\beta_{2,\alpha}}, \tag{4.1}
$$

$$
\mathsf{E}\left[e^{-\alpha\tau_b}1_{\{X_{\tau_b} - b > y\}}\right] = e^{-\eta_1 y}\frac{(\eta_1 - \beta_{1,\alpha})(\beta_{2,\alpha} - \eta_1)}{\eta_1(\beta_{2,\alpha} - \beta_{1,\alpha})}\left[e^{-b\beta_{1,\alpha}} - e^{-b\beta_{2,\alpha}}\right], \qquad \forall y \ge 0, \tag{4.2}
$$

$$
\mathsf{E}\left[e^{-\alpha\tau_b}1_{\{X_{\tau_b} = b\}}\right] = \frac{\eta_1 - \beta_{1,\alpha}}{\beta_{2,\alpha} - \beta_{1,\alpha}} e^{-b\beta_{1,\alpha}} + \frac{\beta_{2,\alpha} - \eta_1}{\beta_{2,\alpha} - \beta_{1,\alpha}} e^{-b\beta_{2,\alpha}}. \tag{4.3}
$$

Proof. Here we focus on the proof for (4.1) and (4.2). Equation (4.3) follows immediately by taking the difference of (4.1) and (4.2) by letting $y = 0$. For notation simplicity, we shall write $\beta_i \equiv \beta_{i,\alpha}$, $i = 1,\ 2$.

Proof of "(4.1)": For any fixed level $b > 0$, define the function u to be

$$u(x) \triangleq \begin{cases} 1 & ; \quad x < b \\ A_1 e^{-\beta_1(b-x)} + B_1 e^{-\beta_2(b-x)} & ; \quad x \geq b \end{cases},$$

where A_1 and B_1 are defined to be the two coefficients in front of the exponential terms in (4.1). Clearly $0 \leq u(x) \leq 1$ for all $x \in (-\infty, \infty)$, because $\beta_1,\ \beta_2 \geq 0$. Note that on the set $\{\tau_b < \infty\}$, $u(X_{\tau_b}) = 1$ since $A_1 + B_1 = 1$. Furthermore, the function u is continuous.

Plugging in this form of u and doing the integration in two regions, $\int_{-\infty}^{\infty} = \int_0^{b-x} + \int_{b-x}^{\infty}$, we have, after some algebra, that, for all $x < b$, $-\alpha u + \mathcal{L}u$ is equal to

$$A_1 e^{-(b-x)\beta_1} f(\beta_1) + B_1 e^{-(b-x)\beta_2} f(\beta_2) - \lambda p e^{-\eta_1(b-x)} \left(A_1 \frac{\eta_1}{\eta_1 - \beta_1} + B_1 \frac{\eta_1}{\eta_1 - \beta_2} - 1 \right), \tag{4.4}$$

where $f(\beta) \triangleq G(\beta) - \alpha$. Since

$$f(\beta_1) = f(\beta_2) = 0, \quad A_1 \frac{\eta_1}{\eta_1 - \beta_1} + B_1 \frac{\eta_1}{\eta_1 - \beta_2} - 1 = 0,$$

we have

$$-\alpha u(x) + \mathcal{L}u(x) = 0, \qquad \forall x < b. \tag{4.5}$$

Because function $u(x)$ is continuous, but not C^1 at $x = b$, we cannot apply Itô formula directly to process $\{e^{-\alpha t} u(X_t);\ t \geq 0\}$. However, it is not difficult to see that there exists a sequence of functions $\{u_n(x);\ n = 1, 2, \cdots\}$ such that: (1) $u_n(x)$ is smooth everywhere, and in particular it belongs to C^2; (2) $u_n(x) = u(x)$ for all $x \leq b$; (3) $u_n(x) = 1 = u(x)$ for all $x \geq b + \frac{1}{n}$; (4) $0 \leq u_n(x) \leq 2$ for all x and n. Clearly, $u_n(x) \to u(x)$ for all x.

It follows from a straightforward calculation that, for $x < b$,

$$\begin{aligned}
\mathcal{L}u_n(x) &= \frac{1}{2}\sigma^2 u_n''(x) + \mu u_n'(x) + \lambda \int_{-\infty}^{\infty} [u_n(x+y) - u_n(x)]\, f_Y(y)\, dy \\
&= \frac{1}{2}\sigma^2 u_n''(x) + \mu u_n'(x) - \lambda u_n(x) + \lambda \int_{-\infty}^{\infty} u_n(x+y) f_Y(y)\, dy \\
&= \frac{1}{2}\sigma^2 u''(x) + \mu u'(x) - \lambda u(x) + \lambda \int_{-\infty}^{b-x} u(x+y) f_Y(y)\, dy \\
&\qquad + \lambda \int_{b-x}^{b-x+\frac{1}{n}} u_n(x+y) f_Y(y)\, dy + \lambda \int_{b-x+\frac{1}{n}}^{\infty} u(x+y) f_Y(y)\, dy \\
&= \alpha u(x) + \lambda \int_{b-x}^{b-x+\frac{1}{n}} u_n(x+y) \cdot f_Y(y)\, dy - \lambda \int_{b-x}^{b-x+\frac{1}{n}} u(x+y) f_Y(y)\, dy,
\end{aligned}$$

thanks to (4.5). Since $|u_n - u| \le 1$ by construction, it follows that,

$$|-\alpha u_n(x) + \mathcal{L}u_n(x)| \le \lambda p \int_{b-x}^{b-x+\frac{1}{n}} \left|u_n(x+y) - u(x+y)\right| \cdot \eta_1 \, dy \le \frac{\lambda p \eta_1}{n} \;\to\; 0, \quad \forall\, x < b \tag{4.6}$$

uniformly in x, as $n \to \infty$. Applying the Itô formula for jump processes (see, e.g., Protter, 1990) to the process $\{e^{-\alpha t} u_n(X_t);\ t \ge 0\}$, we obtain that the process

$$M_t^{(n)} \triangleq e^{-\alpha(t\wedge\tau_b)} u_n\left(X_{t\wedge\tau_b}\right) - \int_0^{t\wedge\tau_b} e^{-\alpha s}(-\alpha u_n(X_s) + \mathcal{L}u_n(X_s))\, ds, \qquad t \ge 0,$$

is a local martingale starting from $M_0^{(n)} = u_n(0) = u(0)$. However,

$$\left|M_t^{(n)}\right| \le 2 + \frac{\lambda p \eta_1}{n} t, \qquad \forall t \ge 0,$$

thanks to (4.6). It follows from the Dominated Convergence Theorem, $\{M_t^{(n)};\ t \ge 0\}$ is actually a martingale. In particular,

$$\mathsf{E} M_t^{(n)} = \mathsf{E}\left[e^{-\alpha(t\wedge\tau_b)} u_n\left(X_{t\wedge\tau_b}\right) - \int_0^{t\wedge\tau_b} e^{-\alpha s}\big(-\alpha u_n(X_s) + \mathcal{L}u_n(X_s)\big)\, ds\right] = u(0),$$

for all $t \ge 0$. Letting $n \to \infty$, It follows from Dominated Convergence Theorem that

$$\lim_{n\to\infty} \mathsf{E}\left[e^{-\alpha(t\wedge\tau_b)} u_n\left(X_{t\wedge\tau_b}\right)\right] = \mathsf{E}\left[e^{-\alpha(t\wedge\tau_b)} u(X_{t\wedge\tau_b})\right],$$

and, thanks to the uniform convergence in (4.6),

$$\lim_{n\to\infty} \mathsf{E}\left[\int_0^{t\wedge\tau_b} e^{-\alpha s}\big(-\alpha u_n(X_s) + \mathcal{L}u_n(X_s)\big)\, ds\right] = \lim_{n\to\infty} \mathsf{E}\left[\int_0^{t\wedge\tau_b -} e^{-\alpha s}\big(-\alpha u_n(X_s) + \mathcal{L}u_n(X_s)\big)\, ds\right] = 0.$$

Therefore, for any $t \ge 0$,

$$u(0) = \mathsf{E}\left[e^{-\alpha(t\wedge\tau_b)} u(X_{t\wedge\tau_b})\right] = \mathsf{E}\left[e^{-\alpha(t\wedge\tau_b)} u(X_{t\wedge\tau_b}) 1_{\{\tau_b < \infty\}}\right] + \mathsf{E}\left[e^{-\alpha t} u(X_t) 1_{\{\tau_b = \infty\}}\right].$$

Now letting $t \to \infty$, we have, thanks to the boundedness of u,

$$u(0) = \mathsf{E}\left[e^{-\alpha\tau_b} u(X_{\tau_b}) 1_{\{\tau_b < \infty\}}\right] = \mathsf{E}\left[e^{-\alpha\tau_b} 1_{\{\tau_b < \infty\}}\right] = \mathsf{E}\left[e^{-\alpha\tau_b}\right],$$

as $u(X_{\tau_b}) = 1$ on set $\{\tau_b < \infty\}$, from which the result follows.

Proof of "(4.2)": it is very similar to the previous proof, so we only give an outline. It suffices to consider the case where $y > 0$, as the case for $y = 0$ follows by letting $y \downarrow 0$. Let $u(x) \triangleq \mathsf{E}^x\left[e^{-\alpha\tau_b} 1_{\{X_{\tau_b} - b > y\}}\right]$, we expect that u satisfies the equation

$$-\alpha u(x) + \mathcal{L}u(x) = 0$$

for all $x < b$, and $u(x) = 1$ if $x \geq b + y$ while $u(x) = 0$ if $x \in [b, b + y)$. This equation can be explicitly solved. Indeed, consider a solution taking form

$$u(x) = \begin{cases} 1 & ; \quad x > b + y \\ 0 & ; \quad b < x \leq b + y \\ A_2 e^{-(b-x)\beta_1} + B_2 e^{-(b-x)\beta_2} & ; \quad x \leq b \end{cases}$$

where constants A_2 and B_2 are yet to be determined. Plug in to obtain that

$$\big(-\alpha u + \mathcal{L} u\big)(x) = A_2 e^{-(b-x)\beta_1} f(\beta_1) + B_2 e^{-(b-x)\beta_2} f(\beta_2) - \lambda p e^{-\eta_1 (b-x)} \left(\frac{A_2 \eta_1}{\eta_1 - \beta_1} + \frac{B_2 \eta_1}{\eta_1 - \beta_2} - e^{-\eta_1 y} \right) = 0.$$

for all $x < b$. Since $f(\beta_1) = f(\beta_2) = 0$, it suffices to choose A_2 and B_2 so that

$$A_2 \frac{\eta_1}{\eta_1 - \beta_1} + B_2 \frac{\eta_1}{\eta_1 - \beta_2} = e^{-\eta_1 y}.$$

Furthermore, the continuity of u at $x = b$ implies that

$$A_2 + B_2 = 0.$$

Solve the equations to obtain A_2 and B_2 ($A_2 = -B_2$), which are exactly the coefficients in (4.2). A similar argument as before yields

$$u(0) = \mathsf{E}\left[e^{-\alpha\tau_b} u(X_{\tau_b}) 1_{\{\tau_b < \infty\}}\right] = \mathsf{E}\left[e^{-\alpha\tau_b} 1_{\{X_{\tau_b} > b+y\}} 1_{\{\tau_b < \infty\}}\right] = \mathsf{E}\left[e^{-\alpha\tau_b} 1_{\{X_{\tau_b} - b > y\}}\right],$$

as $u(X_{\tau_b}) = 1_{\{X_{\tau_b} > b+y\}}$ on the set $\{\tau_b < \infty\}$, from which the proof is terminated. □

Note the following Laplace transform, which is convenient for numerical Laplace inversion,

$$\int_0^\infty e^{-\alpha t} \mathsf{P}(\tau_b \leq t) dt = \frac{1}{\alpha} \int_0^\infty e^{-\alpha t} d\mathsf{P}(\tau_b \leq t) = \frac{1}{\alpha} \mathsf{E}(e^{-\alpha\tau_b}).$$

Remark 4.1. Here some very special features of exponential density functions enable us to explicitly solve the differential equations associated with the Laplace transforms; see, for example, the three-term decomposition in (4.4). For general jump diffusion processes, such explicit solutions might not be available, partly because of the lack of the three-term decomposition.

4.2 Properties

Corollary 4.1. *We have* $\mathsf{P}(\tau_b < \infty) = 1$ *if and only if* $\bar{u} \geq 0$. *Furthermore, if* $\bar{u} \geq 0$, *then*

$$\begin{aligned} \mathsf{P}(X_{\tau_b} - b > y) &= e^{-\eta_1 y} \frac{\beta_2^* - \eta_1}{\beta_2^*} \left[1 - e^{-b\beta_2^*}\right], \quad \forall y \geq 0, \\ \mathsf{P}(X_{\tau_b} = b) &= \frac{\eta_1}{\beta_2^*} + \frac{\beta_2^* - \eta_1}{\beta_2^*} e^{-b\beta_2^*}. \end{aligned}$$

If $\bar{u} < 0$, then

$$\begin{aligned}
\mathsf{P}(\tau_b < \infty) &= \frac{\eta_1 - \beta_1^*}{\eta_1} \cdot \frac{\beta_2^*}{\beta_2^* - \beta_1^*} e^{-b\beta_1^*} + \frac{\beta_2^* - \eta_1}{\eta_1} \cdot \frac{\beta_1^*}{\beta_2^* - \beta_1^*} e^{-b\beta_2^*} < 1, \\
\mathsf{P}(\tau_b < \infty,\ X_{\tau_b} - b > y) &= e^{-\eta_1 y} \frac{(\eta_1 - \beta_1^*)(\beta_2^* - \eta_1)}{\eta_1(\beta_2^* - \beta_1^*)} \left[e^{-b\beta_1^*} - e^{-b\beta_2^*} \right], \quad \forall y \geq 0, \\
\mathsf{P}(\tau_b < \infty,\ X_{\tau_b} = b) &= \frac{\eta_1 - \beta_1^*}{\beta_2^* - \beta_1^*} e^{-b\beta_1^*} + \frac{\beta_2^* - \eta_1}{\beta_2^* - \beta_1^*} e^{-b\beta_2^*}.
\end{aligned}$$

Here β_1^* and β_2^* are defined as in Lemma 3.1.

Proof. By Lemma 3.1, if $\bar{u} \geq 0$, then, as $\alpha \to 0$, $\beta_{1,\alpha} \to 0$ and $\beta_{2,\alpha} \to \beta_2^*$. Thus,

$$\mathsf{P}(\tau_b < \infty) = \lim_{\alpha \to 0} \mathsf{E}[e^{-\alpha \tau_b}] = 1.$$

If $\bar{u} < 0$, then, as $\alpha \to 0$, $\beta_{1,\alpha} \to \beta_1^*$ and $\beta_{2,\alpha} \to \beta_2^*$. The result follows by letting $\alpha \to 0$ in (4.1), (4.2), and (4.3). □

Remark 4.2. It is surprising to see from Theorem 4.1 and Corollary 4.1 that the first passage time τ_b and the overshoot $X_{\tau_b} - b$ are dependent, although Proposition 3.1 shows that they are conditionally independent.

Corollary 4.2. *The expectation of the first passage time is finite, i.e. $\mathsf{E}[\tau_b] < \infty$, if and only if $\bar{u} > 0$. Indeed,*

$$\mathsf{E}[\tau_b] = \left\{ \begin{array}{cc} \frac{1}{\bar{u}} \left[b + \frac{\beta_2^* - \eta_1}{\eta_1 \beta_2^*} (1 - e^{-b\beta_2^*}) \right] & ; \ \ \textit{if} \ \ \bar{u} > 0 \\ +\infty & ; \ \ \textit{if} \ \ \bar{u} \leq 0 \end{array} \right\}.$$

Furthermore, for $\bar{u} < 0$, we have

$$\mathsf{E}\left[\tau_b 1_{\{\tau_b < \infty\}} \right] = C_1 e^{-b\beta_1^*} + C_2 e^{-b\beta_2^*} < \infty,$$

where

$$\begin{aligned}
C_1 &\triangleq \frac{1}{\eta_1(\beta_2^* - \beta_1^*)^2} \left[\frac{\beta_2^*(\beta_2^* - \eta_1) + b\beta_2^*(\eta_1 - \beta_1^*)(\beta_2^* - \beta_1^*)}{G'(\beta_1^*)} + \frac{\beta_1^*(\eta_1 - \beta_1^*)}{G'(\beta_2^*)} \right], \\
C_2 &\triangleq \frac{1}{\eta_1(\beta_2^* - \beta_1^*)^2} \left[\frac{\beta_1^*(\beta_1^* - \eta_1) + b\beta_1^*(\eta_1 - \beta_2^*)(\beta_1^* - \beta_2^*)}{G'(\beta_2^*)} + \frac{\beta_2^*(\eta_1 - \beta_2^*)}{G'(\beta_1^*)} \right].
\end{aligned}$$

See Lemma 3.1 for the definition of (β_1^*, β_2^*).

Proof: To ease exposition, we will use β_i to denote $\beta_{i,\alpha}$. Since the function $\frac{1}{x}(1 - e^{-x})$ is decreasing for $x \in [0, +\infty)$, it follows that with probability one,

$$\frac{1 - e^{-\alpha \tau_b}}{\alpha} 1_{\{\tau_b < \infty\}} \uparrow \tau_b 1_{\{\tau_b < \infty\}} \qquad \text{as} \quad \alpha \downarrow 0.$$

By Monotone Convergence Theorem, we have

$$\mathsf{E}\left[\tau_b 1_{\{\tau_b<\infty\}}\right] = \lim_{\alpha\downarrow 0}\mathsf{E}\left[\frac{1-e^{-\alpha\tau_b}}{\alpha}1_{\{\tau_b<\infty\}}\right] = \lim_{\alpha\downarrow 0}\frac{\mathsf{P}(\tau_b<\infty)-\mathsf{E}e^{-\alpha\tau_b}}{\alpha} = -\lim_{\alpha\to 0}\frac{d}{d\alpha}\mathsf{E}e^{-\alpha\tau_b},$$

where the last equality follows from L'hospital rule. However, it follows from Implicit Function Theorem that

$$\lim_{\alpha\to 0}\frac{d}{d\alpha}\beta_i = \lim_{\alpha\to 0}\frac{1}{G'(\beta_i)} = \frac{1}{G'(\beta_i^*)}.$$

1. For $\bar{u}\geq 0$, we have $\mathsf{P}(\tau_b<\infty)=1$ and $\mathsf{E}[\tau_b 1_{\{\tau_b<\infty\}}]=\mathsf{E}[\tau_b]$. Moreover, in this case, we have $\beta_1\to 0$, $\beta_2\to\beta_2^*$ as $\alpha\to 0$, and $G'(0)=\bar{u}$, according to Lemma 3.1.

2. For $\bar{u}<0$, it is trivial that $\mathsf{E}[\tau_b]=\infty$. Moreover, in this case, $\beta_1\to\beta_1^*$, $\beta_2\to\beta_2^*$ as $\alpha\to 0$, where β_1^* and β_2^* are both positive.

The rest of the proof is a straightforward calculation, and thus is omitted. □

Corollary 4.3. *For any $\alpha>0$ and $\theta<\eta_1$, we have*

$$\mathsf{E}\left[e^{-\alpha\tau_b+\theta X_{\tau_b}}1_{\{\tau_b<\infty\}}\right] = e^{\theta b}\left[c_1 e^{-b\beta_{1,\alpha}} + c_2 e^{-b\beta_{2,\alpha}}\right],$$

where

$$c_1 = \frac{\eta_1-\beta_{1,\alpha}}{\beta_{2,\alpha}-\beta_{1,\alpha}}\cdot\frac{\beta_{2,\alpha}-\theta}{\eta_1-\theta}, \qquad c_2 = \frac{\beta_{2,\alpha}-\eta_1}{\beta_{2,\alpha}-\beta_{1,\alpha}}\cdot\frac{\beta_{1,\alpha}-\theta}{\eta_1-\theta}.$$

Proof. It follows that

$$\begin{aligned}
\mathsf{E}\left[e^{-\alpha\tau_b+\theta X_{\tau_b}}1_{\{\tau_b<\infty\}}\right] &= \mathsf{E}\left[e^{-\alpha\tau_b+\theta X_{\tau_b}}1_{\{X_{\tau_b}=b,\ \tau_b<\infty\}}\right] + e^{\theta b}\mathsf{E}\left[e^{-\alpha\tau_b+\theta(X_{\tau_b}-b)}1_{\{X_{\tau_b}>b,\ \tau_b<\infty\}}\right] \\
&= e^{\theta b}\mathsf{E}\left[e^{-\alpha\tau_b}1_{\{X_{\tau_b}=b,\ \tau_b<\infty\}}\right] + e^{\theta b}\frac{\eta_1}{\eta_1-\theta}\mathsf{E}\left[e^{-\alpha\tau_b}1_{\{X_{\tau_b}>b,\ \tau_b<\infty\}}\right] \\
&= e^{\theta b}\mathsf{E}\left[e^{-\alpha\tau_b}1_{\{X_{\tau_b}=b\}}\right] + e^{\theta b}\frac{\eta_1}{\eta_1-\theta}\mathsf{E}\left[e^{-\alpha\tau_b}1_{\{X_{\tau_b}>b\}}\right],
\end{aligned}$$

where we have used the conditional memoryless property. The claim follows from Theorem 4.1. □

Note that if $\bar{u}\geq 0$, then $\mathsf{P}(\tau_b<\infty)=1$ and Corollary 4.3 implies that

$$\mathsf{E}\left[e^{-\alpha\tau_b+\beta_{1,\alpha}X_{\tau_b}}\right]=1,$$

which can be verified alternatively by applying the optional sampling theorem to the exponential martingale

$$e^{\beta_{1,\alpha}X_t-G(\beta_{1,\alpha})t} = e^{\beta_{1,\alpha}X_t-\alpha t}, \qquad t\geq 0.$$

4.3 Connection with Renewal-Type Integral Equations

We have used martingale and differential equations to derive closed form solutions of the Laplace transforms for the first-passage-time probabilities. Another possible and popular approach to solving the problems, now investigated in this section, is to set up some integral equations by using renewal arguments. For simplicity, we shall only consider the case of overall drift being nonnegative, i.e. $\bar{u} \geq 0$, in which $\tau_b < \infty$ almost surely.

For any $x > 0$, define $P(x)$ as the probability of no overshoot occurs for the first passage time τ_x with $X_0 \equiv 0$, that is

$$P(x) \stackrel{\triangle}{=} \mathsf{P}(X_{\tau_x} = x). \tag{4.7}$$

Proposition 4.1. *$P(x)$ satisfies the following renewal type integral equation:*

$$P(x+y) = P(y)P(x) + \big(1 - P(x)\big) \int_0^y P(y-z) \cdot \eta_1 e^{-\eta_1 z}\, dz,$$

However, the solution to this renewal equation is not unique. Indeed, for every $\xi \geq 0$, the function

$$P_\xi(x) = \frac{\eta_1}{\eta_1 + \xi} + \frac{\xi}{\eta_1 + \xi} e^{-(\eta_1 + \xi)x}$$

satisfies the integral equation with the boundary condition $P_\xi(0) = 1$.

Proof. A renewal argument and the memoryless property of the exponential distribution yield

$$\begin{aligned} P(x+y) &= \mathsf{P}(X_{\tau_{x+y}} = x+y \,\big|\, X_{\tau_x} = x)\mathsf{P}(X_{\tau_x} = x) + \mathsf{P}(X_{\tau_{x+y}} = x+y \,\big|\, X_{\tau_x} > x)\mathsf{P}(X_{\tau_x} > x) \\ &= P(y)P(x) + \big(1 - P(x)\big) \int_0^y P(y-z) \cdot \eta_1 e^{-\eta_1 z}\, dz, \end{aligned}$$

thanks to the fact that τ_b is finite almost surely. Now it remains to check that $P_\xi(x)$ satisfies the integral equation for every $\xi \geq 0$. To this end, note that

$$\int_0^y P_\xi(y-z) \cdot \eta_1 e^{-\eta_1 z} dz = \frac{\eta_1}{\eta_1 + \xi}(1 - e^{-\eta_1 y}) + \frac{\eta_1}{\eta_1 + \xi} e^{-\eta_1 y} - \frac{\eta_1}{\eta_1 + \xi} e^{-(\eta_1 + \xi)y}.$$

It is then very easy to check that

$$(1 - P_\xi(x)) \int_0^y P_\xi(y-z) \cdot \eta_1 e^{-\eta_1 z} dz = \frac{\xi \eta_1}{(\eta_1 + \xi)^2} \left[1 - e^{-(\eta_1+\xi)x} - e^{-(\eta_1+\xi)y} + e^{-(\eta_1+\xi)(x+y)}\right],$$

$$P_\xi(x)P_\xi(y) = \frac{1}{(\eta_1 + \xi)^2}\{\eta_1^2 + \eta_1 \xi e^{-(\eta_1+\xi)x} + \eta_1 \xi e^{-(\eta_1+\xi)y} + \xi^2 e^{-(\eta_1+\xi)(x+y)}\}.$$

Thus,

$$P_\xi(x)P_\xi(y) + (1 - P_\xi(x)) \int_0^y P_\xi(y-z) \cdot \eta_1 e^{-\eta_1 z} dz = \frac{\eta_1}{\eta_1 + \xi} + \frac{\xi}{\eta_1 + \xi} e^{-(\eta_1+\xi)(x+y)} = P_\xi(x+y),$$

and the proof is complete. □

Remark 4.3. Proposition 4.1 shows that, in the presence of two-sided jumps, the renewal-type integral equations may not have unique solutions, mainly because of the difficulty of determining enough boundary conditions based on renewal arguments alone. It is easy to see that $\xi = -P'_\xi(0)$. Indeed, as we have shown in Corollary 4.1, it is possible to use the infinitesimal generator and differential equations to determine ξ. The point here is, however, that the renewal-type integral equations cannot do the job by themselves.

5 Joint Distribution of Jump Diffusion and Its Running Maxima

The following probability

$$\mathsf{P}(X_t \geq a, \max_{0 \leq s \leq t} X_s \geq b) = \mathsf{P}(X_t \geq a, \tau_b \leq t),$$

for some fixed numbers $a \leq b$ and $b > 0$, is useful, for example, in pricing barrier options while the logarithm of the underlying asset price is modeled by a jump diffusion process. In this section, we evaluate the Laplace transform $\int_0^\infty e^{-\alpha t}\mathsf{P}(X_t \geq a, \tau_b \leq t)\,dt$, for all $\alpha > 0$. It turns out that the above Laplace transform has an explicit expression, in terms of Hh functions. We shall first give a brief account of the Hh functions.

5.1 Hh Functions

The Hh functions are defined as

$$(5.1) \qquad \mathrm{Hh}_n(x) \triangleq \int_x^\infty \mathrm{Hh}_{n-1}(y)\,dy = \frac{1}{n!}\int_x^\infty (t-x)^n e^{-\frac{t^2}{2}}\,dt, \qquad n = 0, 1, 2, \cdots;$$

$$(5.2) \qquad \mathrm{Hh}_{-1}(x) \triangleq e^{-\frac{x^2}{2}}, \qquad \mathrm{Hh}_0(x) \triangleq \sqrt{2\pi}\Phi(-x),$$

where $\Phi(x)$ is the cumulative distribution function of the standard normal density. The Hh functions are non-increasing, and have a three term recursion, which is very useful in numerical calculation:

$$(5.3) \qquad \mathrm{Hh}_n(x) = \frac{1}{n}\mathrm{Hh}_{n-2}(x) - \frac{x}{n}\mathrm{Hh}_{n-1}(x); \qquad n \geq 1;$$

for more details, see Abramowitz and Stegun (1972, p. 691).

Introduce the following function

$$(5.4) \qquad H_i(a, b, c; n) \triangleq \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{(\frac{1}{2}c^2 - b)t} t^{n+\frac{i}{2}} \mathrm{Hh}_i\left(c\sqrt{t} + \frac{a}{\sqrt{t}}\right) dt.$$

Here $i \geq -1, n \geq 0$ are both integers and we assume:

(5.5) *Assumption:* a, b, c are arbitrary constants such that $b > 0$ and $c > -\sqrt{2b}$.

For $i \geq 1$, it follows from identity (5.3) that

$$H_i(a,b,c;n) = \frac{1}{i}H_{i-2}(a,b,c;n+1) - \frac{c}{i}H_{i-1}(a,b,c;n+1) - \frac{a}{i}H_{i-1}(a,b,c;n). \tag{5.6}$$

This recursive formula can be used to determine all the values of H_i's, starting from $H_{-1}(a,b,c;n)$ and $H_0(a,b,c;n)$. See Appendix A for details.

5.2 Laplace Transform

Proposition 5.1. *The Laplace transform of the joint distribution is given by*

$$\int_0^\infty e^{-\alpha t}\mathsf{P}(X_t \geq a,\ \tau_b \leq t)\,dt = A\int_0^\infty e^{-\alpha t}\mathsf{P}(X_t \geq a-b)\,dt + B\int_0^\infty e^{-\alpha t}\mathsf{P}(X_t + \xi^+ \geq a-b)\,dt.$$

Here ξ^+ is an independent exponential random variable with rate $\eta_1 > 0$, and

$$A \triangleq \mathsf{E}\left[e^{-\alpha\tau_b}1_{\{X_{\tau_b}=b\}}\right] = \frac{\eta_1 - \beta_{1,\alpha}}{\beta_{2,\alpha}-\beta_{1,\alpha}}e^{-b\beta_{1,\alpha}} + \frac{\beta_{2,\alpha}-\eta_1}{\beta_{2,\alpha}-\beta_{1,\alpha}}e^{-b\beta_{2,\alpha}} \tag{5.7}$$

$$B \triangleq \mathsf{E}\left[e^{-\alpha\tau_b}1_{\{X_{\tau_b}>b\}}\right] = \frac{(\eta_1-\beta_{1,\alpha})(\beta_{2,\alpha}-\eta_1)}{\eta_1(\beta_{2,\alpha}-\beta_{1,\alpha})}\left[e^{-b\beta_{1,\alpha}} - e^{-b\beta_{2,\alpha}}\right]. \tag{5.8}$$

Proof. We need to compute two integrals

$$\mathrm{I} = \int_0^\infty e^{-\alpha t}\mathsf{P}(X_t \geq a, X_{\tau_b} = b, \tau_b \leq t)dt, \quad \mathrm{II} = \int_0^\infty e^{-\alpha t}\mathsf{P}(X_t \geq a, X_{\tau_b} > b, \tau_b \leq t)dt.$$

For the first one,

$$\begin{aligned}
\mathrm{I} &= \int_0^\infty e^{-\alpha t}\int_0^t \mathsf{P}(X_t \geq a, X_{\tau_b} = b, \tau_b \in ds)\,dt \\
&= \int_0^\infty \int_0^t e^{-\alpha t}\mathsf{P}(X_{\tau_b} = b, \tau_b \in ds)\mathsf{P}(X_{t-s} \geq a-b)\,dt \\
&= \int_0^\infty e^{-\alpha s}\mathsf{P}(X_{\tau_b} = b, \tau_b \in ds)\cdot\int_0^\infty e^{-\alpha u}\mathsf{P}(X_u \geq a-b)\,du \\
&= \mathsf{E}\left[e^{-\alpha\tau_b}1_{\{X_{\tau_b}=b\}}\right]\cdot\int_0^\infty e^{-\alpha t}\mathsf{P}(X_t \geq a-b)\,dt,
\end{aligned}$$

where the second equality follows from the strong Markov property, and the third equality follows from the fact that the Laplace transform of convolution is the product of Laplace transforms.

As for the second term, observe that for any $s \in [0,t]$,

$$\mathsf{P}(X_t \geq a,\ X_{\tau_b} > b,\ \tau_b \in ds) = \mathsf{P}(X_{\tau_b} > b,\ \tau_b \in ds)\mathsf{P}(X_{t-s} + \xi^+ \geq a-b),$$

by the conditional memoryless property and the conditional independence (see Proposition 3.1), as well as the strong Markov property; here ξ^+ is some independent exponential random variable with rate η_1. It follows exactly like term (I) that

$$\mathbb{I} = \mathsf{E}\left[e^{-\alpha\tau_b} 1_{\{X_{\tau_b}>b\}}\right] \cdot \int_0^\infty e^{-\alpha t} \mathsf{P}(X_t + \xi^+ \geq a-b)\, dt,$$

from which the proof is completed. □

Theorem 5.1. *The Laplace transform of the joint distribution can be further written as*

$$\begin{aligned}
\int_0^\infty e^{-\alpha t}\mathsf{P}(X_t \geq a,\ \tau_b \leq t)\, dt &= (A+B)\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}H_0(-h,\Upsilon_\alpha,-\frac{\mu}{\sigma};n) \\
&+ e^{h\sigma\eta_1}\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{\lambda^n}{n!}(AP_{n,j}+B\bar{P}_{n,j})\left(\sum_{i=0}^{j-1}(\sigma\eta_1)^i H_i(h,\Upsilon_\alpha,c_+;n)\right) \\
&- e^{-h\sigma\eta_2}\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{\lambda^n}{n!}(AQ_{n,j}+B\bar{Q}_{n,j})\left(\sum_{i=0}^{j-1}(\sigma\eta_2)^i H_i(-h,\Upsilon_\alpha,c_-;n)\right) \\
&+ e^{h\sigma\eta_1}B\sum_{n=1}^{\infty}\sum_{i=0}^{n}\frac{(\lambda p)^n}{n!}(\sigma\eta_1)^i H_i(h,\Upsilon_\alpha,c_+;n) + e^{h\sigma\eta_1}BH_0(h,\Upsilon_\alpha,c_+;0).
\end{aligned}$$

Here

$$\begin{aligned}
P_{n,i} &\triangleq \sum_{j=i}^{n-1}\binom{n}{j}p^j q^{n-j}\binom{n-i-1}{j-i}\left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{j-i}\left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{n-j} \quad \forall 1\leq i\leq n-1, \\
Q_{n,i} &\triangleq \sum_{j=i}^{n-1}\binom{n}{j}q^j p^{n-j}\binom{n-i-1}{j-i}\left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{j-i}\left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{n-j} \quad \forall 1\leq i\leq n-1,
\end{aligned}$$

while $P_{n,n} \triangleq p^n$ *and* $Q_{n,n} \triangleq q^n$*;*

$$\bar{P}_{n,1} \triangleq \sum_{i=1}^{n} Q_{n,i}\left(\frac{\eta_2}{\eta_1+\eta_2}\right)^i; \qquad \bar{P}_{n,i} = P_{n,i-1}, \qquad \forall\ 2\leq i\leq n+1,$$

$$\bar{Q}_{n,i} \triangleq \sum_{j=i}^{n}\binom{n}{j}q^j p^{n-j}\binom{n-i}{j-i}\left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{j-i}\left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{n-j+1}, \qquad \forall\ 1\leq i\leq n;$$

(5.9) $$c_+ \triangleq \sigma\eta_1 + \frac{\mu}{\sigma}, \quad c_- \triangleq \sigma\eta_2 - \frac{\mu}{\sigma}, \quad \Upsilon_\alpha \triangleq \alpha + \lambda + \frac{\mu^2}{2\sigma^2}, \quad h \triangleq \frac{b-a}{\sigma},$$

and A and B are given by equations(5.7) and (5.8).

The proof of this theorem is long and is given in Appendix B.

Remark 5.1. All the parameters involved in H functions in Theorem 5.1 satisfy the assumption (5.5).

Remark 5.2. It is easy to derive the corresponding result for $\mathsf{P}(X_t \le -a,\ \tilde{\tau}_{-b} \le t)$, $a \le b$, $b > 0$, where $\tilde{\tau}_{-b} := \inf\{t \ge 0 :\ X_t \le -b\}$, More precisely, one only needs to do the following changes in Theorem 5.1: $p \to q$, $q \to p$, $\beta_{1,\alpha} \to \beta_{3,\alpha}$, $\eta_1 \to \eta_2$, $\eta_2 \to \eta_1$, and $\beta_{2,\alpha} \to \beta_{4,\alpha}$.

6 Laplace Inversion and Numerical Examples

Since the solutions of the first passage times are given in terms of Laplace transforms, numerical inversion of Laplace transforms becomes necessary. To do this, we shall use the Gaver-Stehfest algorithm. The reason is that among all the Laplace inversion algorithms, to our best knowledge, the Gaver-Stehfest is the only one that does the inversion on the real line; all others perform the calculation in the complex domain, which are unsuitable for our purpose as the Laplace transforms in our case involve finding the roots $\beta_{1,\alpha}$ and $\beta_{2,\alpha}$. For a survey of Laplace inversion algorithms, see the paper by Abate and Whitt (1992).

The algorithm is very simple. For any bounded real-valued function $f(\cdot)$ defined on $[0,\infty)$ that is continuous at t,

$$f(t) = \lim_{n\to\infty} \tilde{f}_n(t),$$

where

$$\tilde{f}_n(t) = \frac{\ln(2)}{t}\frac{(2n)!}{n!(n-1)!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\hat{f}\left((n+k)\frac{\ln(2)}{t}\right), \tag{6.1}$$

and $\hat{f}$ is the Laplace transform of f, i.e. $\hat{f}(\alpha) = \int_0^\infty e^{-\alpha t} f(t)dt$. To speed up the convergence, an n-point Richardson extrapolation can be used. More precisely, $f(t)$ can be approximated by $f_n^*(t)$, for large n, where

$$f_n^*(t) = \sum_{k=1}^{n} w(k,n)\tilde{f}_k(t),$$

and the extrapolation weights $w(k,n)$ are given by

$$w(k,n) = (-1)^{n-k}\frac{k^n}{k!(n-k)!}. \tag{6.2}$$

Numerically, we find that it is better to ignore the first few initial calculations of $\tilde{f}_k$. As a result the algorithm approximates $f(t)$ by $f_n^*(t)$, where

$$f_n^*(t) = \sum_{k=1}^{n} w(k,n)\tilde{f}_{k+B}(t),$$

with $\tilde{f}$ and w given by (6.1) and (6.2), and $B \geq 0$ is the initial burning-out number (typically equals to 2 or 3).

Main advantages of the Gaver-Stehfest algorithm are: (a) it is very easy to program (several lines of code will do the job); (b) it converges very fast; as we will see, the algorithm typically converges nicely even for n between 5 and 10; (c) it is stable (i.e. small perturbation of initial inputs will not lead to a dramatic change of final results), if high accuracy computation is used. A main disadvantage of the algorithm is that it needs high accuracy, as both $\tilde{f}_n$ and the weights $w(k,n)$ involve factorials and alternative $+/-$ signs. In our numerical examples, typically an accuracy of 30-80 digits is needed. However, in many software packages (e.g. "Mathematica") one can specify arbitrary accuracy, and in standard programming languages (e.g. C++) subroutines for high precision calculation are available. So this is not a big problem.

It is easy to compute the marginal and joint distributions of the first passages times for the double exponential jump diffusion process by using the Laplace transform formulae given in Sections 4 and 5, in conjunction with the Gaver-Stehfest algorithm. As a numerical illustration, we shall present two examples, one is to compute $\mathsf{P}(\tau_b \leq t)$, and the other $\mathsf{P}(\tau_b \leq t,\ X_t \geq a)$, for $b = 0.3$, $a = 0.2$, and $t = 1$. The parameters, which are chosen to reflect those in typical finance applications, for the double exponential jump diffusion are $\mu = \pm 0.1$, $\sigma = 0.2$, $p = 0.5$, $\eta_1 = 1/0.02$, $\eta_2 = 1/0.03$, and $\lambda = 3$. To make a comparison with the Monte Carlo simulation, we also use $\lambda = 0.01$, so that the results may be compared with the limiting Brownian motion case ($\lambda = 0$); the formulae for the first passage times of Brownian motion can be found in many textbooks; e.g. Karlin and Taylor (1975).

All the computations are done on a Pentium 400 PC. The initial burning-out number used in all calculations is $B = 2$. Also, in calculating $\mathsf{P}(\tau_b \leq t,\, X_t \geq a)$, we truncate the Poisson sum after the 10th term, as additional numerical calculations suggest that the error involves in the truncation is less than 10^{-6}. The reason why the calculation of $\mathsf{P}(\tau_b \leq t,\, X_t \geq a)$ takes longer time is that it requires of computing the functions H_i recursively and "Mathematica" is slow in terms of recursive calculation.

To speed up the simulation, binomial apprixomation is used to simulate the Poisson processes. Note that the Monte Carlo simulation is biased and slow, due to two sources of errors: random sampling error and systematic discretization bias. It is quite possible to signicantly reduce the random sampling error here (thus the width of the confidence intervals) by using some variance reduction techniques, such as control variates and importance sampling (suitable for the case of $\bar{u} < 0$). The systematic discretization bias, resulting from approximating a continuous time process by a discrete time process in simulation, is, however, very difficult to be reduced; in the examples

n		$\mathsf{P}(\tau_b \le t)$		$\mathsf{P}(\tau_b \le t,\ X_t \ge a)$	
		$\lambda = 0.01$	$\lambda = 3$	$\lambda = 0.01$	$\lambda = 3$
1		0.34669	0.33472	0.30266	0.28114
2		0.30818	0.29912	0.27062	0.25673
3		0.28211	0.27521	0.24940	0.23849
4		0.26880	0.26313	0.23886	0.22886
5		0.26328	0.25819	0.23464	0.22507
6		0.26136	0.25649	0.23324	0.22393
7		0.26078	0.25599	0.23285	0.22367
8		0.26063	0.25587	0.23277	0.22363
9		0.26060	0.25585	0.23275	0.22362
10		0.26060	0.25584	0.23275	0.22362
Total CPU Time		1.26 sec	1.76 sec	4.53 min	4.61 min
Brownian Motion Case		0.26061	N.A.	0.23278	N.A.
Monte Carlo Simulation					
200 Points	point est.	0.248	0.244	0.226	0.218
CPU Time: 15 min	95% C.I.	(0.241, 0.255)	(0.236, 0.252)	(0.220, 0.232)	(0.211, 0.225)
2000 Points	point est.	0.254	0.251	0.227	0.220
CPU Time: 1 hr 20 min	95% C.I.	(0.247, 0.261)	(0.244, 0.258)	(0.220, 0.234)	(0.214, 0.226)

Table 1: Positive Overall Drift $\bar{u} > 0$ ($\mu = 0.1$).

n		$\mathsf{P}(\tau_b \le t)$		$\mathsf{P}(\tau_b \le t,\ X_t \ge a)$	
		$\lambda = 0.01$	$\lambda = 3$	$\lambda = 0.01$	$\lambda = 3$
1		0.07737	0.07884	0.04762	0.04626
2		0.06878	0.07096	0.04591	0.04558
3		0.06296	0.06562	0.04455	0.04480
4		0.05999	0.06289	0.04376	0.04428
5		0.05876	0.06176	0.04340	0.04404
6		0.05833	0.06137	0.04328	0.04397
7		0.05820	0.06126	0.04325	0.04396
8		0.05817	0.06123	0.04325	0.04396
9		0.05816	0.06122	0.04325	0.04397
10		0.05816	0.06122	0.04325	0.04397
Total CPU Time		1.20 sec	1.81 sec	4.49 min	4.67 min
Brownian Motion Case		0.05815	N.A.	0.04324	N.A.
Monte Carlo Simulation					
200 Points	point est.	0.055	0.056	0.042	0.043
CPU Time: 15 min	95% C.I.	(0.051, 0.059)	(0.052, 0.060)	(0.038, 0.046)	(0.040, 0.046)
2000 Points	point est.	0.057	0.059	0.043	0.044
CPU Time: 1 hr 20 min	95% C.I.	(0.053, 0.061)	(0.055, 0.063)	(0.040, 0.046)	(0.041, 0.047)

Table 2: Negative Overall Drift $\bar{u} < 0$ ($\mu = -0.1$).

given above, it makes the calculation from the simulation biased low. Even in the Brownian motion case, because of the presence of boundary, the discretization bias is very significant, resulting in a surprisingly slow rate of convergence for simulating the first passage time, both theoretically and numerically; see Asmussen, Glynn, and Pitman (1995) (they show that the discretization error has an order 1/2, which is much slower than the order 1 convergence for simulation without the boundary; 16,000 points are suggested for a Brownian motion with $\mu = -1, \sigma = 1$ and time $t = 8$). In the presence of jumps, the discretization bias is even more serious, especially for large t or λ. This explains the large bias in our simulation results.

A Appendix. Computation of H_i function

We have defined in (5.4) the following function

$$H_i(a,b,c;n) \triangleq \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{(\frac{1}{2}c^2-b)t} t^{n+\frac{i}{2}} \mathrm{Hh}_i\left(c\sqrt{t} + \frac{a}{\sqrt{t}}\right) dt$$

for integers $i \geq -1, n \geq 0$. *Throughout this section, we assume that (5.5) always holds, that is,* $b > 0, c > -\sqrt{2b}$. The following recursion formula holds:

$$H_i(a,b,c;n) = \frac{1}{i} H_{i-2}(a,b,c;n+1) - \frac{c}{i} H_{i-1}(a,b,c;n+1) - \frac{a}{i} H_{i-1}(a,b,c;n).$$

Therefore, it suffices to evaluate $H_{-1}(a,b,c;n)$ and $H_0(a,b,c;n)$, both of which can be calculated explicitly.

Lemma A.1. *If $a \neq 0$, then for all integers $n \geq 0$ we have*

$$H_{-1}(a,b,c;n) = e^{-ac-\sqrt{2a^2b}} \sqrt{\frac{1}{2b}} \left(\sqrt{\frac{a^2}{2b}}\right)^n \cdot \sum_{j=0}^{n} \frac{(-n)_j (n+1)_j}{j!(-2\sqrt{2a^2b})^j},$$

and for all integers $n \leq -1$

$$H_{-1}(a,b,c;n) = e^{-ac-\sqrt{2a^2b}} \sqrt{\frac{1}{2b}} \left(\sqrt{\frac{a^2}{2b}}\right)^n \cdot \sum_{j=0}^{-n-1} \frac{(-n)_j (n+1)_j}{j!(-2\sqrt{2a^2b})^j},$$

where $(a)_j \triangleq a(a+1)\cdots(a+j-1)$. If $a = 0$, then for all integers $n \geq 0$ we have

$$H_{-1}(0,b,c;n) = \frac{(2n)!}{n!(4b)^n} \sqrt{\frac{1}{2b}},$$

and for all integers $n \leq -1$ we have $H_{-1}(0,b,c;n) = +\infty$.

Proof: We shall prove the case of $a \neq 0$ first. Since $\mathrm{Hh}_{-1}(x) = e^{-\frac{x^2}{2}}$, by definition

$$H_{-1}(a,b,c;n) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{(\frac{1}{2}c^2-b)t} t^{n-\frac{1}{2}} e^{-\frac{1}{2}\left(c\sqrt{t}+\frac{a}{\sqrt{t}}\right)^2} dt = \frac{1}{\sqrt{2\pi}} e^{-ac} \int_0^\infty e^{-\left(bt+\frac{a^2}{2t}\right)} t^{n-\frac{1}{2}} dt.$$

Recall the so-called modified Bessel Function of the third kind (Bateman, 1953, p. 5), $K_\nu(x)$, which has an integral representation (Bateman, 1954, p. 146) as

$$\frac{\alpha^\nu}{2} \int_0^\infty e^{-\frac{z}{2}\left(t+\frac{\alpha^2}{t}\right)} \cdot \frac{1}{t^{\nu+1}} dt = K_\nu(\alpha z)$$

for arbitrary constants ν and $\alpha > 0, z > 0$. It is easy to show that

$$H_{-1}(a,b,c;n) = \sqrt{\frac{2}{\pi}} e^{-ac} \left(\sqrt{\frac{a^2}{2b}}\right)^{n+\frac{1}{2}} K_{-(n+\frac{1}{2})}(\sqrt{2a^2 b}).$$

However, the modified Bessel function $K_\nu(x)$ has property that

$$K_\nu(x) = K_{-\nu}(x), \qquad \forall \nu,$$

$$K_{n+\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \sum_{j=0}^{n} \frac{(-n)_j (n+1)_j}{j!(-2x)^j}, \qquad \forall n \geq 0;$$

see Bateman (1953, p. 5 and p. 10). The result follows.

Now consider the case of $a = 0$. We have, by definition, that

$$H_{-1}(0,b,c;n) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-bt} t^{n-\frac{1}{2}} dt.$$

For $n \leq -1$, the integral is obviously $+\infty$. For $n \geq 0$, this integral is the Laplace transform of $t^{n-\frac{1}{2}}$, which can be found in many integral tables, and we have

$$H_{-1}(0,b,c;n) = \frac{1}{\sqrt{2\pi}} \frac{(2n)!}{n!(4b)^n} \sqrt{\frac{\pi}{b}} = \frac{(2n)!}{n!(4b)^n} \sqrt{\frac{1}{2b}}.$$

This completes the proof. □

The following lemma gives the value of H_0 in terms of H_{-1}.

Lemma A.2. *Suppose $n \geq 0$ is an integer.*

1. *If $b = \frac{1}{2}c^2$, we have*

$$H_0(a,b,c;n) = \frac{c}{2(n+1)} H_{-1}(a,b,c;n+1) - \frac{a}{2(n+1)} H_{-1}(a,b,c;n).$$

2. If $b \neq \frac{1}{2}c^2$, $a > 0$, we have

$$H_0(a,b,c;n) = \frac{n!}{(b-\frac{1}{2}c^2)^{n+1}} \sum_{i=0}^{n} \frac{(b-\frac{1}{2}c^2)^i}{i!} \left(\frac{a}{2} H_{-1}(a,b,c;i-1) - \frac{c}{2} H_{-1}(a,b,c;i) \right).$$

3. If $b \neq \frac{1}{2}c^2$, $a < 0$, we have

$$H_0(a,b,c;n) = \frac{n!}{(b-\frac{1}{2}c^2)^{n+1}} + \frac{n!}{(b-\frac{1}{2}c^2)^{n+1}} \sum_{i=0}^{n} \frac{(b-\frac{1}{2}c^2)^i}{i!} \left(\frac{a}{2} H_{-1}(a,b,c;i-1) - \frac{c}{2} H_{-1}(a,b,c;i) \right).$$

4. If $b \neq \frac{1}{2}c^2$, $a = 0$, we have

$$H_0(0,b,c;n) = \frac{n!}{2(b-\frac{1}{2}c^2)^{n+1}} - \frac{n!}{(b-\frac{1}{2}c^2)^{n+1}} \sum_{i=0}^{n} \frac{(b-\frac{1}{2}c^2)^i}{i!} \frac{c}{2} H_{-1}(0,b,c;i).$$

Proof: It follows from the definition of Hh function (5.1) that

$$\frac{d}{dx}\mathrm{Hh}_n(x) = -\mathrm{Hh}_{n-1}(x), \qquad n = 0,1,2,\cdots$$

For $b = \frac{1}{2}c^2$ (i.e. $c = \sqrt{2b} > -\sqrt{2b}$), since $\mathrm{Hh}_0(x) = \sqrt{2\pi}\Phi(-x)$, we have

$$\begin{aligned}
H_0(a,b,c;n) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty t^n \mathrm{Hh}_0\left(c\sqrt{t} + \frac{a}{\sqrt{t}}\right) dt \\
&= \frac{1}{\sqrt{2\pi}} \frac{t^{n+1}}{n+1} \mathrm{Hh}_0\left(c\sqrt{t} + \frac{a}{\sqrt{t}}\right)\Big|_0^\infty + \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{t^{n+1}}{n+1} \mathrm{Hh}_{-1}\left(c\sqrt{t} + \frac{a}{\sqrt{t}}\right) \cdot \left(\frac{c}{2\sqrt{t}} - \frac{a}{2t\sqrt{t}}\right) dt \\
&= \frac{c}{2(n+1)} \frac{1}{\sqrt{2\pi}} \int_0^\infty t^{n+\frac{1}{2}} \mathrm{Hh}_{-1}\left(c\sqrt{t} + \frac{a}{\sqrt{t}}\right) dt - \frac{a}{2(n+1)} \frac{1}{\sqrt{2\pi}} \int_0^\infty t^{n-\frac{1}{2}} \mathrm{Hh}_{-1}\left(c\sqrt{t} + \frac{a}{\sqrt{t}}\right) dt \\
&= \frac{c}{2(n+1)} H_{-1}(a,b,c;n+1) - \frac{a}{2(n+1)} H_{-1}(a,b,c;n).
\end{aligned}$$

For $b \neq \frac{1}{2}c^2$, we have the following elementary identity

$$\frac{d}{dt}\left(\frac{-n!}{(b-\frac{1}{2}c^2)^{n+1}} e^{(\frac{1}{2}c^2-b)t} \sum_{i=0}^{n} \frac{(b-\frac{1}{2}c^2)^i t^i}{i!} \right) = e^{(\frac{1}{2}c^2-b)t} t^n,$$

and equality (note $c > -\sqrt{2b}$)

$$\frac{1}{\sqrt{2\pi}} \left(\frac{-n!}{(b-\frac{1}{2}c^2)^{n+1}} e^{(\frac{1}{2}c^2-b)t} \sum_{i=0}^{n} \frac{(b-\frac{1}{2}c^2)^i t^i}{i!} \right) \cdot \mathrm{Hh}_0\left(c\sqrt{t} + \frac{a}{\sqrt{t}}\right)\Big|_0^\infty = \begin{cases} 0 & ; \text{ if } a > 0 \\ \frac{n!}{(b-\frac{1}{2}c^2)^{n+1}} & ; \text{ if } a < 0 \\ \frac{n!}{2(b-\frac{1}{2}c^2)^{n+1}} & ; \text{ if } a = 0 \end{cases}$$

The rest of the proof is simply integration by parts, and is thus omitted. □

B Appendix. Proof of Theorem 5.1

The proof relies on four lemmas, of which the first two can be found in Kou (1999). The third one can be proved by a modification of Proposition 1 in Kou (1999). Thus, we only give a proof for the last lemma.

Lemma B.1. *Suppose* $\{\xi_1, \xi_2, \cdots\}$ *is a sequence of i.i.d. exponential random variables with rate* η, *and* Z *is a normal random variable with distribution* $N(0, \sigma^2)$. *For* $n \geq 1$,

1. *The tail probability of random variable* $Z + \sum_{i=1}^n \xi_i$ *is given by*

$$\mathsf{P}\left(Z + \sum_{i=1}^n \xi_i \geq x\right) = \frac{(\sigma\eta)^n}{\sigma\sqrt{2\pi}} e^{\frac{1}{2}(\sigma\eta)^2} I_{n-1}(x; -\eta, -\frac{1}{\sigma}, -\sigma\eta).$$

2. *The tail probability of random variable* $Z - \sum_{i=1}^n \xi_i$ *is given by*

$$\mathsf{P}\left(Z - \sum_{i=1}^n \xi_i \geq x\right) = \frac{(\sigma\eta)^n}{\sigma\sqrt{2\pi}} e^{\frac{1}{2}(\sigma\eta)^2} I_{n-1}(x;\ \eta,\ \frac{1}{\sigma}, -\sigma\eta).$$

Here the function I_n *is defined as*

$$I_n(c; \alpha, \beta, \delta) \triangleq \int_c^\infty e^{\alpha x} Hh_n(\beta x - \delta)\, dx. \tag{B.1}$$

Lemma B.2. *If* $\beta > 0$, $\alpha \neq 0$:

$$I_n(c; \alpha, \beta, \delta) = -\frac{e^{\alpha c}}{\alpha} \sum_{i=0}^n \left(\frac{\beta}{\alpha}\right)^{n-i} \mathrm{Hh}_i(\beta c - \delta) + \left(\frac{\beta}{\alpha}\right)^{n+1} \frac{\sqrt{2\pi}}{\beta} e^{\frac{\alpha\delta}{\beta} + \frac{\alpha^2}{2\beta^2}} \Phi\left(-\beta c + \delta + \frac{\alpha}{\beta}\right).$$

If $\beta < 0$, $\alpha < 0$:

$$I_n(c; \alpha, \beta, \delta) = -\frac{e^{\alpha c}}{\alpha} \sum_{i=0}^n \left(\frac{\beta}{\alpha}\right)^{n-i} \mathrm{Hh}_i(\beta c - \delta) - \left(\frac{\beta}{\alpha}\right)^{n+1} \frac{\sqrt{2\pi}}{\beta} e^{\frac{\alpha\delta}{\beta} + \frac{\alpha^2}{2\beta^2}} \Phi\left(\beta c - \delta - \frac{\alpha}{\beta}\right).$$

If $\beta > 0$, $\alpha = 0$:

$$I_n(c; 0, \beta, \delta) = \int_c^\infty \mathrm{Hh}_n(\beta x - \delta)\, dx = \frac{1}{\beta} \mathrm{Hh}_{n+1}(\beta c - \delta).$$

Lemma B.3. *For any fixed* $t \geq 0$, *conditioning on* $N_t = n$, $n \geq 1$, X_t *has a decomposition in distribution*

$$X_t \quad \sim \quad \left\{ \begin{array}{llll} \mu t + Z + \sum_{i=1}^j \xi_i^+ & \textit{with probability} & P_{n,j} & ; \quad j = 1, 2, \cdots, n \\ \mu t + Z - \sum_{i=1}^j \xi_i^- & \textit{with probability} & Q_{n,j} & ; \quad j = 1, 2, \cdots, n \end{array} \right\},$$

and $X_t + \xi^+$ *has a decomposition in distribution*

$$X_t + \xi^+ \quad \sim \quad \left\{ \begin{array}{llll} \mu t + Z + \sum_{i=1}^j \xi_i^+ & \textit{with probability} & \bar{P}_{n,j} & ; \quad j = 1, 2, \cdots, n+1 \\ \mu t + Z - \sum_{i=1}^j \xi_i^- & \textit{with probability} & \bar{Q}_{n,j} & ; \quad j = 1, 2, \cdots, n \end{array} \right\},$$

where Z is a normal random variable with distribution $N(0,\sigma^2 t)$, and $\{\xi_i^+; i \geq 1\}$, $\{\xi_i^-; i \geq 1\}$ are i.i.d. exponential random variables with rates η_1 and η_2, respectively.

Lemma B.4. *We have*

$$\text{(B.2)} \qquad \int_0^\infty e^{-\left(\alpha+\lambda\right)t} t^n \frac{\left(\sigma\sqrt{t}\eta_1\right)^j}{\sigma\sqrt{2\pi t}} e^{\frac{1}{2}(\eta_1\sigma)^2 t} I_{j-1}\left(-h\sigma-\mu t; -\eta_1, -\frac{1}{\sigma\sqrt{t}}, -\eta_1\sigma\sqrt{t}\right) dt$$
$$= H_0(-h,\Upsilon_\alpha,-\frac{\mu}{\sigma};n) + e^{\eta_1 h\sigma}\sum_{i=0}^{j-1}(\sigma\eta_1)^i H_i(h,\Upsilon_\alpha,c_+;n),$$

$$\text{(B.3)} \qquad \int_0^\infty e^{-\left(\alpha+\lambda\right)t} t^n \frac{\left(\sigma\sqrt{t}\eta_2\right)^j}{\sigma\sqrt{2\pi t}} e^{\frac{1}{2}(\eta_2\sigma)^2 t} I_{j-1}\left(-h\sigma-\mu t;\ \eta_2,\ \frac{1}{\sigma\sqrt{t}}, -\eta_2\sigma\sqrt{t}\right) dt$$
$$= H_0(-h,\Upsilon_\alpha,-\frac{\mu}{\sigma};n) - e^{-h\sigma\eta_2}\sum_{i=0}^{j-1}(\sigma\eta_2)^i H_i(-h,\Upsilon_\alpha,c_-;n).$$

Proof. In (B.2) the I_{j-1} function, $j \geq 1$, can be split into two parts by Lemma B.2:

$$\begin{aligned} I_{j-1} &= \frac{\sigma\sqrt{2\pi t}}{\left(\sigma\sqrt{t}\eta_1\right)^j} e^{-\frac{1}{2}(\eta_1\sigma)^2 t}\Phi\left(\frac{h}{\sqrt{t}}+\frac{\mu}{\sigma}\sqrt{t}\right) + \frac{e^{\eta_1 h\sigma}}{\eta_1} e^{\mu\eta_1 t}\sum_{i=0}^{j-1}\left(\frac{1}{\sigma\sqrt{t}\eta_1}\right)^{j-i-1}\mathrm{Hh}_i\left(\frac{h}{\sqrt{t}}+c_+\sqrt{t}\right) \\ &=: (A)+(B). \end{aligned}$$

The term (A) will contribute

$$\begin{aligned} &\int_0^\infty e^{-\left(\alpha+\lambda\right)t} t^n \frac{\left(\sigma\sqrt{t}\eta_1\right)^j}{\sigma\sqrt{2\pi t}} e^{\frac{1}{2}(\eta_1\sigma)^2 t}\cdot\frac{\sigma\sqrt{2\pi t}}{\left(\sigma\sqrt{t}\eta_1\right)^j} e^{-\frac{1}{2}(\eta_1\sigma)^2 t}\Phi\left(\frac{h}{\sqrt{t}}+\frac{\mu}{\sigma}\sqrt{t}\right) dt \\ &= \int_0^\infty e^{-\left(\alpha+\lambda\right)t} t^n \frac{1}{\sqrt{2\pi}}\mathrm{Hh}_0\left(-\frac{h}{\sqrt{t}}-\frac{\mu}{\sigma}\sqrt{t}\right) dt = H_0(-h,\Upsilon_\alpha,-\frac{\mu}{\sigma};n). \end{aligned}$$

The contribution from (B) is

$$\begin{aligned} &\sum_{i=0}^{j-1}\int_0^\infty e^{-\left(\alpha+\lambda\right)t} t^n \frac{\left(\sigma\sqrt{t}\eta_1\right)^j}{\sigma\sqrt{2\pi t}} e^{\frac{1}{2}(\eta_1\sigma)^2 t}\cdot\frac{e^{\eta_1 h\sigma}}{\eta_1} e^{\mu\eta_1 t}\left(\frac{1}{\sigma\sqrt{t}\eta_1}\right)^{j-i-1}\mathrm{Hh}_i\left(\frac{h}{\sqrt{t}}+c_+\sqrt{t}\right) dt \\ &= e^{\eta_1 h\sigma}\sum_{i=0}^{j-1}(\sigma\eta_1)^i\int_0^\infty e^{\left(-\alpha-\lambda+\mu\eta_1+\frac{1}{2}\sigma^2\eta_1^2\right)t} t^{n+\frac{i}{2}}\cdot\frac{1}{\sqrt{2\pi}}\mathrm{Hh}_i\left(\frac{h}{\sqrt{t}}+c_+\sqrt{t}\right) dt \\ &= e^{\eta_1 h\sigma}\sum_{i=0}^{j-1}(\sigma\eta_1)^i H_i(h,\Upsilon_\alpha,c_+;n). \end{aligned}$$

In (B.3), the function I_{j-1} can be split similarly into two terms:

$$\begin{aligned} I_{j-1} &= \frac{\sigma\sqrt{2\pi t}}{\left(\sigma\sqrt{t}\eta_2\right)^j} e^{-\frac{1}{2}(\eta_2\sigma)^2 t}\Phi\left(\frac{h}{\sqrt{t}}+\frac{\mu}{\sigma}\sqrt{t}\right) - \frac{e^{-h\eta_2\sigma}}{\eta_2} e^{-\mu\eta_2 t}\sum_{i=0}^{j-1}\left(\frac{1}{\sigma\sqrt{t}\eta_2}\right)^{j-i-1}\mathrm{Hh}_i\left(-\frac{h}{\sqrt{t}}+c_-\sqrt{t}\right) \\ &=: (A)+(B). \end{aligned}$$

Here the term (A) will contribute

$$\int_0^\infty e^{-\left(\alpha+\lambda\right)t} t^n \frac{\left(\sigma\sqrt{t}\eta_2\right)^j}{\sigma\sqrt{2\pi t}} e^{\frac{1}{2}(\eta_2\sigma)^2 t} \cdot \frac{\sigma\sqrt{2\pi t}}{\left(\sigma\sqrt{t}\eta_2\right)^j} e^{-\frac{1}{2}(\eta_2\sigma)^2 t} \Phi\left(\frac{h}{\sqrt{t}} + \frac{\mu}{\sigma}\sqrt{t}\right) dt$$
$$= \int_0^\infty e^{-\left(\alpha+\lambda\right)t} t^n \frac{1}{\sqrt{2\pi}} \mathrm{Hh}_0\left(-\frac{h}{\sqrt{t}} - \frac{\mu}{\sigma}\sqrt{t}\right) dt = H_0(-h, \Upsilon_\alpha, -\frac{\mu}{\sigma}; n);$$

and the term (B) will contribute

$$-\sum_{i=0}^{j-1} \int_0^\infty e^{-\left(\alpha+\lambda\right)t} t^n \frac{\left(\sigma\sqrt{t}\eta_2\right)^j}{\sigma\sqrt{2\pi t}} e^{\frac{1}{2}(\eta_2\sigma)^2 t} \cdot \frac{e^{-h\sigma\eta_2}}{\eta_2} e^{-\mu\eta_2 t} \left(\frac{1}{\sigma\sqrt{t}\eta_2}\right)^{j-i-1} \mathrm{Hh}_i\left(-\frac{h}{\sqrt{t}} + c_-\sqrt{t}\right) dt$$
$$= -e^{-h\sigma\eta_2} \sum_{i=0}^{j-1} (\sigma\eta_2)^i \int_0^\infty e^{\left(-\alpha-\lambda-\mu\eta_2+\frac{1}{2}\sigma^2\eta_2^2\right)t} t^{n+\frac{i}{2}} \cdot \frac{1}{\sqrt{2\pi}} \mathrm{Hh}_i\left(-\frac{h}{\sqrt{t}} + c_-\sqrt{t}\right) dt$$
$$= -e^{-h\sigma\eta_2} \sum_{i=0}^{j-1} (\sigma\eta_2)^i H_i(-h, \Upsilon_\alpha, c_-; n),$$

from which the result follows. □.

Proof of Theorem 5.1. From Proposition 5.1 the Laplace transform of the joint distribution is given by

$$\int_0^\infty e^{-\alpha t} \mathsf{P}(X_t \ge a,\ \tau_b \le t)\, dt = A \int_0^\infty e^{-\alpha t} \mathsf{P}(X_t \ge a - b)\, dt + B \int_0^\infty e^{-\alpha t} \mathsf{P}(X_t + \xi^+ \ge a)\, dt,$$

where ξ^+ is an independent exponential distribution with rate η_1.

Let us first consider the first term. By conditioning on N_t, it follows from Lemmas B.1 and B.3 that

$$\begin{aligned}
&\mathsf{P}(X_t \ge a - b) = e^{-\lambda t} \cdot \Phi\left(\frac{\mu t - (a-b)}{\sigma\sqrt{t}}\right) \\
&\quad + \sum_{n=1}^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{j=1}^n P_{n,j} \frac{\left(\sigma\sqrt{t}\eta_1\right)^j}{\sigma\sqrt{2\pi t}} e^{\frac{1}{2}(\eta_1\sigma)^2 t} \cdot I_{j-1}\left(a - b - \mu t; -\eta_1, -\frac{1}{\sigma\sqrt{t}}, -\eta_1\sigma\sqrt{t}\right) \\
&\quad + \sum_{n=1}^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{j=1}^n Q_{n,j} \frac{\left(\sigma\sqrt{t}\eta_2\right)^j}{\sigma\sqrt{2\pi t}} e^{\frac{1}{2}(\eta_2\sigma)^2 t} \cdot I_{j-1}\left(a - b - \mu t;\ \eta_2,\ \frac{1}{\sigma\sqrt{t}}, -\eta_2\sigma\sqrt{t}\right),
\end{aligned} \tag{B.4}$$

Using the identity $\mathrm{Hh}_0(x) = \sqrt{2\pi}\Phi(-x)$ and the definition of H function (5.4), it is easy to check that the first term in (B.4) will contribute

$$\int_0^\infty e^{-(\alpha+\lambda)t} \frac{1}{\sqrt{2\pi}} \mathrm{Hh}_0\left(\frac{-\mu t + (a-b)}{\sigma\sqrt{t}}\right) dt = H_0(\frac{a-b}{\sigma}, \alpha + \lambda + \frac{\mu^2}{2\sigma^2}, -\frac{\mu}{\sigma}; 0) = H_0(-h, \Upsilon_\alpha, -\frac{\mu}{\sigma}; 0),$$

in the notation of (5.9). The second term in (B.4) will contribute

$$\sum_{n=1}^\infty \sum_{j=1}^n \frac{\lambda^n}{n!} P_{n,j} \int_0^\infty e^{-\left(\alpha+\lambda\right)t} t^n \frac{\left(\sigma\sqrt{t}\eta_1\right)^j}{\sigma\sqrt{2\pi t}} e^{\frac{1}{2}(\eta_1\sigma)^2 t} I_{j-1}\left(-h\sigma - \mu t; -\eta_1, -\frac{1}{\sigma\sqrt{t}}, -\eta_1\sigma\sqrt{t}\right) dt$$

(note $a-b=-h\sigma$). The third term in (B.4) will contribute the following

$$\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{\lambda^n}{n!}Q_{n,j}\int_0^\infty e^{-\left(\alpha+\lambda\right)t}t^n\frac{\left(\sigma\sqrt{t}\eta_2\right)^j}{\sigma\sqrt{2\pi t}}e^{\frac{1}{2}(\eta_2\sigma)^2t}I_{j-1}\left(-h\sigma-\mu t;\ \eta_2,\ \frac{1}{\sigma\sqrt{t}},-\eta_2\sigma\sqrt{t}\right)dt.$$

Therefore, we have by Lemma B.4

$$\begin{aligned}
&\int_0^\infty e^{-\alpha t}\mathsf{P}(X_t\geq a-b)\,dt=H_0(-h,\Upsilon_\alpha,-\frac{\mu}{\sigma};0)+\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{\lambda^n}{n!}(P_{n,j}+Q_{n,j})H_0(-h,\Upsilon_\alpha,-\frac{\mu}{\sigma};n)\\
&\quad+e^{h\sigma\eta_1}\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{\lambda^n}{n!}P_{n,j}\sum_{i=0}^{j-1}(\sigma\eta_1)^iH_i(h,\Upsilon_\alpha,c_+;n)-e^{-h\sigma\eta_2}\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{\lambda^n}{n!}Q_{n,j}\sum_{i=0}^{j-1}(\sigma\eta_2)^iH_i(-h,\Upsilon_\alpha,c_-;n)\\
=&\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}H_0(-h,\Upsilon_\alpha,-\frac{\mu}{\sigma};n)+e^{h\sigma\eta_1}\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{\lambda^n}{n!}P_{n,j}\sum_{i=0}^{j-1}(\sigma\eta_1)^iH_i(h,\Upsilon_\alpha,c_+;n)\\
&\quad-e^{-h\sigma\eta_2}\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{\lambda^n}{n!}Q_{n,j}\sum_{i=0}^{j-1}(\sigma\eta_2)^iH_i(-h,\Upsilon_\alpha,c_-;n),
\end{aligned}$$

where the last equality follows from the fact that $\sum_{j=1}^n(P_{n,j}+Q_{n,j})=1$.

It remains to evaluate the Laplace transform $\int_0^\infty e^{-\alpha t}\mathsf{P}(X_t+\xi^+\geq a-b)\,dt$, where ξ^+ is an independent exponential random variable with rate η_1. Conditioning on $N_t=0$, clearly $X_t+\xi^+\sim\ \mu t+Z+\xi^+$, where Z is a normal random variable with distribution $N(0,\sigma^2t)$. Therefore, by conditioning on N_t, we have, via Lemmas B.3 and B.1, that

$$\begin{aligned}
&\mathsf{P}(X_t+\xi^+\geq a-b)=e^{-\lambda t}I_0\left(a-b-\mu t,-\eta_1,-\frac{1}{\sigma\sqrt{t}},-\eta_1\sigma\sqrt{t}\right)\cdot\frac{\eta_1}{\sqrt{2\pi}}e^{\frac{1}{2}\sigma^2\eta_1^2t}\\
&\quad+\sum_{n=1}^{\infty}e^{-\lambda t}\frac{(\lambda t)^n}{n!}\sum_{j=1}^{n+1}\bar{P}_{n,j}\frac{\left(\sigma\sqrt{t}\eta_1\right)^j}{\sigma\sqrt{2\pi t}}e^{\frac{1}{2}(\eta_1\sigma)^2t}\cdot I_{j-1}\left(a-b-\mu t;-\eta_1,-\frac{1}{\sigma\sqrt{t}},-\eta_1\sigma\sqrt{t}\right)\\
&\quad+\sum_{n=1}^{\infty}e^{-\lambda t}\frac{(\lambda t)^n}{n!}\sum_{j=1}^{n}\bar{Q}_{n,j}\frac{\left(\sigma\sqrt{t}\eta_2\right)^j}{\sigma\sqrt{2\pi t}}e^{\frac{1}{2}(\eta_2\sigma)^2t}\cdot I_{j-1}\left(a-b-\mu t;\ \eta_2,\ \frac{1}{\sigma\sqrt{t}},-\eta_2\sigma\sqrt{t}\right).
\end{aligned}\tag{B.5}$$

The first term in (B.5) is

$$\begin{aligned}
&e^{-\lambda t}I_0(-h\sigma-\mu t;-\eta_1,-\frac{1}{\sigma\sqrt{t}},-\eta_1\sigma\sqrt{t})\cdot\frac{\eta_1}{\sqrt{2\pi}}e^{\frac{1}{2}\sigma^2\eta_1^2t}\\
=&\ e^{-\lambda t}\frac{\eta_1}{\sqrt{2\pi}}e^{\frac{1}{2}\sigma^2\eta_1^2t}\cdot\left\{\frac{e^{\eta_1(h\sigma+\mu t)}}{\eta_1}Hh_0(\frac{h\sigma+\mu t}{\sigma\sqrt{t}}+\eta_1\sigma\sqrt{t})\right.\\
&\quad\left.+(\frac{1}{\sigma\eta_1\sqrt{t}})\sigma\sqrt{t}\sqrt{2\pi}\cdot\exp\{(-\eta_1\sigma\sqrt{t})\sigma\eta_1\sqrt{t}+\frac{1}{2}\eta_1^2\sigma^2t\}\cdot\Phi(\frac{h\sigma+\mu t}{\sigma\sqrt{t}}+\eta_1\sigma\sqrt{t}-\eta_1\sigma\sqrt{t})\right\}\\
=&\ e^{-\lambda t}\left\{\frac{1}{\sqrt{2\pi}}e^{\frac{1}{2}\sigma^2\eta_1^2t}e^{\eta_1(h\sigma+\mu t)}Hh_0\left(\frac{h}{\sqrt{t}}+(\frac{\mu}{\sigma}+\eta_1\sigma)\sqrt{t}\right)+\Phi(\frac{h}{\sqrt{t}}+\frac{\mu}{\sigma}\sqrt{t})\right\}.
\end{aligned}$$

Thus, its Laplace transform is given by

$$\int_0^\infty e^{-\alpha t}e^{-\lambda t}\left\{\frac{1}{\sqrt{2\pi}}e^{\frac{1}{2}\sigma^2\eta_1^2 t}e^{\eta_1(h\sigma+\mu t)}Hh_0\left(\frac{h}{\sqrt{t}}+(\frac{\mu}{\sigma}+\eta_1\sigma)\sqrt{t}\right)+\frac{1}{\sqrt{2\pi}}Hh_0(-\frac{h}{\sqrt{t}}-\frac{\mu}{\sigma}\sqrt{t})\right\}dt.$$

Since

$$\begin{aligned}
&\int_0^\infty e^{-(\alpha+\lambda)t}\frac{1}{\sqrt{2\pi}}e^{\frac{1}{2}\sigma^2\eta_1^2 t}e^{\eta_1(h\sigma+\mu t)}Hh_0\left(\frac{h}{\sqrt{t}}+(\frac{\mu}{\sigma}+\eta_1\sigma)\sqrt{t}\right)dt\\
=\ & e^{\eta_1 h\sigma}\frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\{(-\alpha-\lambda+\frac{1}{2}\sigma^2\eta_1^2+\eta_1\mu)t\}Hh_0\left(\frac{h}{\sqrt{t}}+(\frac{\mu}{\sigma}+\eta_1\sigma)\sqrt{t}\right)dt\\
=\ & e^{\eta_1 h\sigma}H_0(h,\Upsilon_\alpha,c_+;0),
\end{aligned}$$

and

$$\int_0^\infty e^{-(\alpha+\lambda)t}\frac{1}{\sqrt{2\pi}}Hh_0(-\frac{h}{\sqrt{t}}-\frac{\mu}{\sigma}\sqrt{t})dt=H_0(-h,\Upsilon_\alpha,-\frac{\mu}{\sigma};0),$$

the contribution of the first term in (B.5) to the Laplace transform is given by

$$e^{\eta_1 h\sigma}H_0(h,\Upsilon_\alpha,c_+;0)+H_0(-h,\Upsilon_\alpha,-\frac{\mu}{\sigma};0).$$

The second term in (B.5) will contribute to the Laplace transform

$$\sum_{n=1}^\infty\sum_{j=1}^{n+1}\frac{\lambda^n}{n!}\bar{P}_{n,j}\int_0^\infty e^{-\left(\alpha+\lambda\right)t}t^n\frac{\left(\sigma\sqrt{t}\eta_1\right)^j}{\sigma\sqrt{2\pi t}}e^{\frac{1}{2}(\eta_1\sigma)^2 t}I_{j-1}\left(-h\sigma-\mu t;-\eta_1,-\frac{1}{\sigma\sqrt{t}},-\eta_1\sigma\sqrt{t}\right)dt$$

(note $a-b=-h\sigma$). The third term in (B.5) will contribute

$$\sum_{n=1}^\infty\sum_{j=1}^{n}\frac{\lambda^n}{n!}\bar{Q}_{n,j}\int_0^\infty e^{-\left(\alpha+\lambda\right)t}t^n\frac{\left(\sigma\sqrt{t}\eta_2\right)^j}{\sigma\sqrt{2\pi t}}e^{\frac{1}{2}(\eta_2\sigma)^2 t}I_{j-1}\left(-h\sigma-\mu t;\ \eta_2,\ \frac{1}{\sigma\sqrt{t}},-\eta_2\sigma\sqrt{t}\right)dt.$$

Therefore, we have, by Lemma B.4,

$$\begin{aligned}
&\int_0^\infty e^{-\alpha t}\mathsf{P}(X_t+\xi^+\geq a-b)\,dt\\
=\ & e^{\eta_1 h\sigma}H_0(h,\Upsilon_\alpha,c_+;0)+H_0(-h,\Upsilon_\alpha,-\frac{\mu}{\sigma};0)+\sum_{n=1}^\infty\frac{\lambda^n}{n!}(\sum_{j=1}^{n+1}\bar{P}_{n,j}+\sum_{j=1}^{n}\bar{Q}_{n,j})H_0(-h,\Upsilon_\alpha,-\frac{\mu}{\sigma};n)\\
&+e^{h\sigma\eta_1}\sum_{n=1}^\infty\sum_{j=1}^{n+1}\frac{\lambda^n}{n!}\bar{P}_{n,j}\sum_{i=0}^{j-1}(\sigma\eta_1)^iH_i(h,\Upsilon_\alpha,c_+;n)-e^{-h\sigma\eta_2}\sum_{n=1}^\infty\sum_{j=1}^{n}\frac{\lambda^n}{n!}\bar{Q}_{n,j}\sum_{i=0}^{j-1}(\sigma\eta_2)^iH_i(-h,\Upsilon_\alpha,c_-;n)\\
=\ & \sum_{n=0}^\infty\frac{\lambda^n}{n!}H_0(-h,\Upsilon_\alpha,-\frac{\mu}{\sigma};n)+e^{h\sigma\eta_1}\sum_{n=1}^\infty\sum_{j=1}^{n}\frac{\lambda^n}{n!}\bar{P}_{n,j}\sum_{i=0}^{j-1}(\sigma\eta_1)^iH_i(h,\Upsilon_\alpha,c_+;n)\\
&-e^{-h\sigma\eta_2}\sum_{n=1}^\infty\sum_{j=1}^{n}\frac{\lambda^n}{n!}\bar{Q}_{n,j}\sum_{i=0}^{j-1}(\sigma\eta_2)^iH_i(-h,\Upsilon_\alpha,c_-;n),\\
&+e^{\eta_1 h\sigma}H_0(h,\Upsilon_\alpha,c_+;0)+e^{h\sigma\eta_1}\sum_{n=1}^\infty\frac{\lambda^n}{n!}p^n\sum_{i=0}^{n}(\sigma\eta_1)^iH_i(h,\Upsilon_\alpha,c_+;n),
\end{aligned}$$

where the last equality follows from the facts that $\sum_{j=1}^{n+1}\bar{P}_{n,j}+\sum_{j=1}^{n}\bar{Q}_{n,j}=1$, and $\bar{P}_{n,n+1}=p^n$. $\square$

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