Capacity Expansion with Exponential Jump Diffusion Processes*

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Abstract

This paper studies an irreversible capacity expansion problem where the industry demand is described by a double exponential jump diffusion process. Formally, we consider the following optimization problem

$$\sup_X \mathsf{E} \int_0^\infty e^{-rt} \left(D_t H(X_t)\, dt - k\, dX_t \right)$$

over all adapted, non-decreasing process $X = (X_t)$. Here we understand X_t as the cumulative capital investment up to time t (it is non-decreasing since the investment is assumed to be irreversible). $D = (D_t)$ is the exogenous industry demand process, which is modeled by a double exponential jump diffusion. The profit flow is assumed to have rate $D \cdot H(X)$ for some concave function H, while k denotes the cost of unit capital. The goal is choosing an capital investment strategy X so as to maximize the discounted overall profit, net the cost of investing.

We explicitly solve the problem when H is assumed to be the Cobb-Douglas production function, that is, $H(x) = x^\alpha$ for some $\alpha \in (0, 1)$. Throughout the paper, a slightly different (but essentially the same) optimization criterion is adopted, for the sake of technical convenience.

Key words: Double exponential jump diffusion, variational inequality, singular control, capacity expansion.
AMS 1991 subject classifications: Primary 90A16, 93E20; secondary 60G44, 60J75.

1 Introduction

Capacity expansion, or incremental investment problems, have been studied by many authors; a very partial list include [3, 4, 11, 12, 13]. Some discussions on the assumption of irreversibility in capital choice can be found in an early publication [2], while a comprehensive treatment of irreversible investment can be found in [5]. Reversible investment problems under similar setup have also received a fair amount of attention; see [1, 16].

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The model of a firm's irreversible incremental investment choice (capacity expansion) can be informally described as follows: the firm has the right to invest in the industry by purchasing capitals. The investment is irreversible in the sense that the existing capitals cannot be sold. The rate of the profit flow from the investment depends on two factors: (1) the current size of the existing capital – the more the capital, the bigger the output rate, whence more profit; (2) the price of the unit output, which is closely related to the industry demand and the size of the output itself – the bigger the demand, or the smaller the output size, the higher the price. By judiciously choosing a capacity expansion strategy, the firm wants to maximize the discounted profit of the investment over a long time horizon, net the cost for purchasing the capitals.

Mathematically, denote by X the size of the capitals in place, which will give a flow of output with rate $Q = G(X)$, the unit price of the output is $P = D \cdot F(Q)$ where D is the industry demand. Hence capital of size X will generate a profit flow with rate

$$PQ = D \cdot F(Q) \cdot G(X) = D \cdot F(G(X)) \cdot G(X) := D \cdot H(X).$$

The function H is usually assumed to be concave, representing the *diminishing return to capital*. If we assume that unit price of the capital is constant k, then the optimization problem is

$$\sup_X \int_0^\infty e^{-rt} \left(D_t H(X_t)\, dt - k\, dX_t \right);$$

here the positive constant r is the discount factor. Explicit solutions are available for very general concave function H, under the classical assumption that the demand $D = (D_t)$ is a geometric Brownian motion; see [5] for more details.

The geometric Brownian motion can be understood as the net effect of many small noises, thanks to the central limit theorem. However, it is not hard to conceive that jump diffusion process might serve as a better model for the industry demand, since some events, e.g. technology breakthrough, or change of government policy, might change the industry demand drastically. But of course, the introduction of the jumps will increase the difficulty of analysis.

In this paper we assume the demand $D = (D_t)$ to be a double exponential jump diffusion process. The double exponential distribution of the jump sizes allow us to find explicit solutions for one of the most important classes of concave functions H; that is, the Cobb-Douglas function $H(x) = x^{1-\alpha}$ for some $\alpha \in (0, 1)$. It is very difficult, if not impossible, to obtain explicit solutions beyond this generality. As a side-note, this type of jump diffusions have recently been used to model the stock prices, and the double exponential jump sizes also make possible for explicit option prices (or rather, their Laplace transforms); see e.g. [6, 8, 9].

Remark 1. In general, the term Cobb-Douglas function refers to a function $H(d, x)$ of both the demand level d and the capital x taking form:

$$H(d, x) = d^\beta\, x^{1-\alpha}, \qquad \alpha, \beta \in (0, 1).$$

Observing that if (D_t) is a double exponential jump diffusion process then so is $(D_t)^\beta$, our analysis can be easily extended to cover this general case.

The paper is organized as follows. Section 2 gives the set up of the problem, and a slightly different optimization criterion is brought up to circumvent technical inconveniences. Section 3 solves the associated variational inequality and provide a verification theorem. The optimal policy turns out to be singular: there are an "investment" region and an "inactive" region, and the optimal policy is to purchase as much capital as needed to prevent the state processes from exiting the "inactive" region. Some of the more technical proofs are collected in the appendix.

2 The capacity expansion problem

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}; \mathsf{P})$ where the filtration $\mathbb{F} = (\mathcal{F}_t)$ satisfies the usual conditions. We assume that the probability space is rich enough to carry a non-negative *demand process* $D = (D_t)$ such that $Y_t \doteq \log(D_t)$ is a jump diffusion process. More precisely,

$$\log(D_t) := Y_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Z_i; \qquad Y_0 \equiv y \in \mathbb{R}. \tag{2.1}$$

Here μ and $\sigma > 0$ are both constants; $W = (W_t, \mathcal{F}_t)$ is a standard Brownian motion; $N = (N_t, \mathcal{F}_t)$ is a right-continuous Poisson process with intensity λ; the jump sizes $Z = (Z_1, Z_2, \cdots)$ are indepent and identically distributed; all the randomness W, N and Z are assumed to be independent. The set of admissible control process is defined as

$$\mathcal{A}(x) \doteq \{X = (X_t, \mathcal{F}_t);\ X \text{ is adapted, right-continuous, non-decreasing, with } X_{0-} \equiv x\}; \tag{2.2}$$

here $X_{0-} = x$ is the initial capital in place. The optimization problem is then introduced as

$$V(x, y) \doteq \sup_{X \in \mathcal{A}(x)} \limsup_{T \to \infty} \mathsf{E} \int_0^T e^{-rt} \left(e^{Y_t} H(X_t)\, dt - k dX_t \right) := \sup_{X \in \mathcal{A}(x)} J(X; y); \tag{2.3}$$

here r (discount factor), k (cost per unit of capital) are positive constants, and H is a nonegative, strictly concave function (production function). A commonly adopted product function is the so-called Cobb-Douglas production function as in (2.5) below. Also see Remark 3 for the discussion on the optimization criterion (2.3).

Throughout this paper, we should assume that the jump sizes (Z_i) are iid double exponential random variables with density

$$f_Z(z) = p \cdot \eta_1 e^{-\eta_1 z} 1_{\{z \geq 0\}} + q \cdot \eta_2 e^{\eta_2 z} 1_{\{z < 0\}}; \qquad p > 0,\ q > 0,\ p + q = 1;\ \eta_1 > 1, \quad \eta_2 > 0. \tag{2.4}$$

Here the assumption $\eta_1 > 1$ is to gurantee that $\mathsf{E}(D_t) = \mathsf{E}[e^{Y_t}]$ is finite for all t. The concave function H is taken to be the Cobb-Douglas production function of form

$$H(x) \doteq x^{1-\alpha}; \qquad \alpha \in (0, 1). \tag{2.5}$$

We also define the following function

$$f(\beta) \doteq -(r + \lambda) + \frac{1}{2}\sigma^2 \beta^2 + \mu\beta + \lambda \left(\frac{p\eta_1}{\eta_1 - \beta} + \frac{q\eta_2}{\eta_2 + \beta} \right), \qquad \beta \in \mathbb{R}. \tag{2.6}$$

It is easy to show that the equation $f(\beta) = 0$ admits exactly two positive solutions (β_1, β_2) with $\beta_1 < \eta_1 < \beta_2$. We should adopt the following assumption throughout the paper.

$$\textbf{Assumption:} \qquad \frac{1}{\alpha} < \beta_1. \tag{2.7}$$

Without this assumption, V could be infinity; see Remark 6 for more details. We arrive at the following inequalities immediately

$$1 < \frac{1}{\alpha} < \beta_1 < \eta_1 < \beta_2. \tag{2.8}$$

It is also easy to check that under this assumption

$$f(1) < 0, \qquad f\left(\frac{1}{\alpha}\right) < 0. \tag{2.9}$$

Remark 2. One can, of course, assume that Z's have some general distribution or H is some general concave function. The corrsponding variational inequality and verification theorem can still be established. However, it is not clear how to obtain explicit solutions with such generality.

Remark 3. Unlike the optimization criterion

$$V(x,y) \doteq \sup_{X\in\mathcal{A}(x)} \mathsf{E}\int_0^\infty e^{-rt}\left(e^{Y_t}H(X_t)\,dt - kdX_t\right) \tag{2.10}$$

often appearing in the economics literature, the version (2.3) adopted here is well defined for every admissible control $X \in \mathcal{A}(x)$ (see the proof of the verification theorem) and is the one commonly used in the control theoy literature. The well-definedness of the expectation in (2.10) for an arbitrary $X \in \mathcal{A}(x)$, however, is questionable. Furthermore, even though (2.10) can still be adopted as the optimization criterion if we are willing to restrict the admissible control processes to those that make (2.10) meaningful, the analysis will be more technical inconvenient, because of the improper integral and the integrand of mixed signs. Finally, (2.3) and (2.10) would formally yield the same variational inequality, and for the optimal capacity choice X^* we obtain, the lim sup in (2.3) is indeed a true limit, and

$$J(X^*;y) = \lim_{T\to\infty} \mathsf{E}\int_0^T e^{-rt}\left(e^{Y_t}H(X_t^*)\,dt - kdX_t^*\right) = \mathsf{E}\int_0^\infty e^{-rt}\left(e^{Y_t}H(X_t^*)\,dt - kdX_t^*\right).$$

See Remark 5 for more details on these equalities.

Remark 4. The dynamics of process Y is unaffected by the control process X. The infinitisimal generator is given as

$$\mathcal{L}u(x,y) = \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial u}{\partial y} + \lambda\int_{\mathbb{R}}\left[u(x,y+z) - u(x,y)\right]f_Z(z)\,dz \tag{2.11}$$

for all twice continuously differentiable functions u. An intuitive explaination for the last term is that the Poisson process has probability λdt to have a jump in a small time interval of length dt, and when a jump occurs, the jump size will have density $f_Z(z)$.

3 Variational inequality and its solution

For every admissible control process X, we associate with the expected discounted profit

$$J(X;y) \doteq \limsup_{T\to\infty} \mathsf{E}\int_0^T e^{-rt}\left(e^{Y_t}H(X_t)\,dt - kdX_t\right)$$

The objective is to maximize $J(X;y)$ over all $X \in \mathcal{A}(x)$ where $X_{0-} \equiv x$.

We will proceed heuristically, in order to obtain the variational inequality associated with the value function V. It follows from Dynamic Programming Principle that the process

$$U_t \doteq \int_0^t e^{-rs}\left(e^{Y_s}H(X_s)\,ds - k\,dX_s\right) + e^{-rt}V(X_t,Y_t)$$

is indeed a supermartingale. Assuming the value function V is twice continuously differentiable, the generalized Itô formula (see [7, 14, 15] for more details) yields

$$dU_t = dM_t + e^{-rt}\left[(-rV+\mathcal{L}V)(X_t,Y_t) + e^{Y_t}H(X_t)\right]\,dt + e^{-rt}\left[V(X_t+\triangle X_t,Y_t) - V(X_t,Y_t) - k\triangle X_t\right],$$

where $M = \{M_t;\ t\geq 0\}$ is some local martingale. Since U is a supermartingale, we expect that

$$-rV + \mathcal{L}V + e^y H(x) \leq 0, \qquad V_x(x,y) \leq k; \qquad \forall\, x \geq 0,\ y \in \mathbb{R}.$$

However, at any time the decision maker can either buy more capital or not, the value function V should satisfy the dynamic programming equation

$$\max\left\{-rV + \mathcal{L}V + e^y H(x),\ V_x - k\right\} = 0, \qquad \forall\, x \geq 0,\ y \in \mathbb{R}.$$

Furthermore, it is reasonable to expect the optimal policy to be such that the bigger the demand Y, the more the capital X should be put in place.

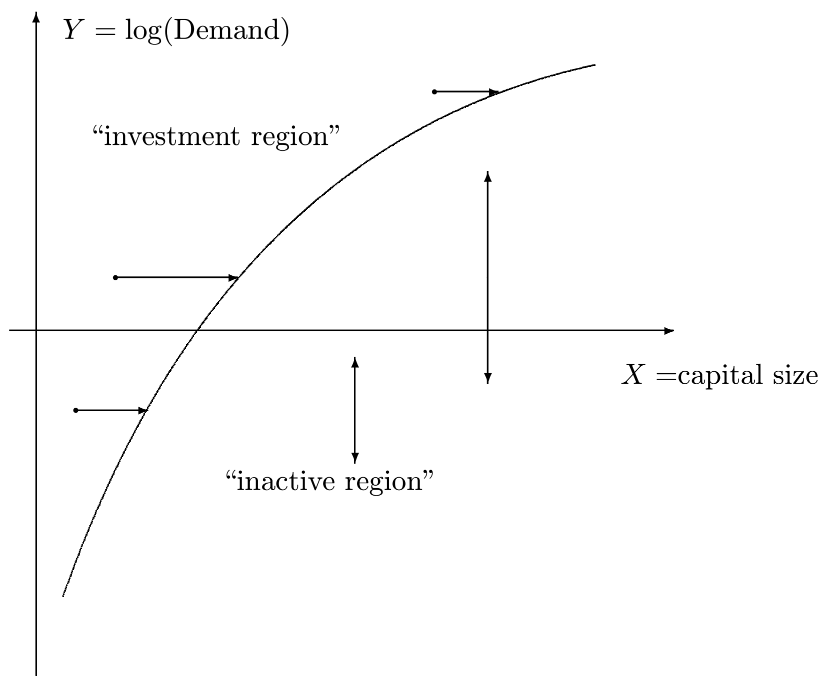

Figure 1.

We obtain the following variational inequality from these heuristic arguments.

Variational Inequality: Find a non-negative, twice continuously differentiable function $v: \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$, and an increasing function $y = g(x)$ such that

$$
\begin{array}{llrcll}
(3.1) & & v_x(x,y) & = & k; & \text{in region } \{(x,y);\ y \geq g(x)\} \\
(3.2) & & -rv + \mathcal{L}v + e^y H(x) & = & 0; & \text{in region } \{(x,y);\ y < g(x)\} \\
(3.3) & & v_x(x,y) & < & k; & \text{in region } \{(x,y);\ y < g(x)\} \\
(3.4) & & -rv + \mathcal{L}v + e^y H(x) & < & 0; & \text{in region } \{(x,y);\ y \geq g(x)\}
\end{array}
$$

This variational inequality can be solved explicitly. It turns out that the solution we obtained is indeed the value function, and the optimal capital policy can be described as follows.

Optimal Capital Policy: Do not increase the capital as long as (X,Y) is in the "inactive" region $\{(x,y);\ y < g(x)\}$. However, when (X,Y) is in the "investment" region $\{(x,y);\ y \geq g(x)\}$, immediately buy as much capital as necessary in order not to exit the region $\{(x,y);\ y \leq g(x)\}$.

The optimal control process is a singular one, with possible discontinuities at $t = 0$ or at times when the Poisson process gives a jump; see Figure 1.

The rest of the section is devoted to solving the variational inequality and proving the verification theorem.

3.1 Solution to the variational inequality

In this section, we solve the variational inequality (3.1)-(3.4). We start with the following guess of the value function and the free boundary, which is motivated by the results in [5] and [10].

Suppose that in the "inactive" region

$$v(x,y) = c_1 x^{p_1} e^{\beta_1 y} + c_2 x^{p_2} e^{\beta_2 y} + c_3 x^{1-\alpha} e^y; \qquad \forall\ x \geq 0,\ y \leq g(x).$$

Here $\beta_{1,2}$ are the two positive solutions to equation $f(\beta) = 0$, and $(c_1, c_2, c_3; p_1, p_2)$ are constants yet to be determined. Equation (3.1) immediately yields the value of v in the "investment" region

$$v(x,y) = -k\left(g^{-1}(y) - x\right) + V\left(g^{-1}(y); y\right); \qquad \forall\ x \geq 0,\ y > g(x).$$

We also guess that the free boundary takes form

$$y = g(x) = \log\big(cx^\gamma\big); \qquad \forall\ x \geq 0,$$

where c and γ are undetermined constants.

It follows from equation (3.1) again and the smooth-fit-principle that, for $y = g(x)$, we have

$$
\begin{aligned}
k \ \equiv\ & v_x(x,y) = c_1 p_1 x^{p_1 - 1} e^{\beta_1 y} + c_2 p_2 x^{p_2 - 1} e^{\beta_2 y} + c_3 (1-\alpha) x^{-\alpha} e^y \\
=\ & c_1 p_1 c^{\beta_1} x^{p_1 - 1 + \gamma\beta_1} + c_2 p_2 c^{\beta_2} x^{p_2 - 1 + \gamma\beta_2} + c_3 (1-\alpha) c x^{-\alpha+\gamma}; \qquad \forall\ x \geq 0.
\end{aligned}
$$

Therefore, we must have $\gamma = \alpha$ and $p_i = 1 - \gamma\beta_i = 1 - \alpha\beta_i$, $i = 1, 2$. We arrive at the following form of solution.

$$
(3.5)\quad v(x,y) = \begin{cases} c_1 x^{1-\alpha\beta_1} e^{\beta_1 y} + c_2 x^{1-\alpha\beta_2} e^{\beta_2 y} + c_3 x^{1-\alpha} e^y & ; \quad \forall\ x \geq 0,\ y \leq g(x) = \log\left(cx^\alpha\right) \\ -k\left(\left(c^{-1}e^y\right)^{\frac{1}{\alpha}} - x\right) + V\left(\left(c^{-1}e^y\right)^{\frac{1}{\alpha}}; y\right) & ; \quad \forall\ x \geq 0,\ y > g(x) = \log\left(cx^\alpha\right) \end{cases};
$$

Four constants $(c_1, c_2, c_3; c)$ remain unknown.

For all $\{(x,y);\ y \le g(x)\}$, straightforward calculation shows that

$$\begin{aligned}
-rv + \mathcal{L}v &= -rv + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial y^2} + \mu \frac{\partial v}{\partial y} + \lambda \int_{\mathbb{R}} \big[v(x, y+z) - v(x,y) \big] f(z)\, dz \\
&= -(r+\lambda) v + \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial y^2} + \mu \frac{\partial v}{\partial y} + \lambda \int_{-\infty}^{0} v(x, y+z) \cdot q\eta_2 e^{\eta_2 z}\, dz \\
&\quad + \lambda \int_{0}^{\log(cx^\alpha)-y} v(x,y+z) \cdot p\eta_1 e^{-\eta_1 z}\, dz + \lambda \int_{\log(cx^\alpha)-y}^{\infty} v(x,y+z) \cdot p\eta_1 e^{-\eta_1 z}\, dz \\
&= c_1 x^{1-\alpha\beta_1} e^{\beta_1 y} \cdot f(\beta_1) + c_2 x^{1-\alpha\beta_2} e^{\beta_2 y} \cdot f(\beta_2) + c_3 x^{1-\alpha} e^{y} \cdot f(1) \\
&\quad + c^{-\eta_1} \frac{\lambda p \eta_1}{\alpha\eta_1 - 1} x^{1-\alpha\eta_1} e^{\eta_1 y} \left[c^{\beta_1} c_1 \cdot \frac{1-\alpha\beta_1}{\eta_1 - \beta_1} + c^{\beta_2} c_2 \cdot \frac{1-\alpha\beta_2}{\eta_1-\beta_2} + c c_3 \cdot \frac{1-\alpha}{\eta_1 - 1} - \frac{k}{\eta_1} \right].
\end{aligned}$$

It follows from equation (3.2) and $f(\beta_1) = f(\beta_2) = 0$ that

$$c_3 f(1) + 1 = 0 \tag{3.6}$$

$$c^{\beta_1} c_1 \cdot \frac{1-\alpha\beta_1}{\eta_1-\beta_1} + c^{\beta_2} c_2 \cdot \frac{1-\alpha\beta_2}{\eta_1-\beta_2} + c c_3 \cdot \frac{1-\alpha}{\eta_1-1} - \frac{k}{\eta_1} = 0. \tag{3.7}$$

The other two equations required to solve for $(c_1, c_2, c_3; c)$ come from the smooth fit conditions across the free boundary , that is,

$$v_x(x,y) \equiv k; \qquad v_{xx}(x,y) \equiv 0; \qquad \forall\ x \ge 0,\ y = g(x) = \log(cx^\alpha).$$

It is easy to check that they are equivalent to

$$c^{\beta_1} c_1 \cdot (1-\alpha\beta_1) + c^{\beta_2} c_2 \cdot (1-\alpha\beta_2) + c c_3 \cdot (1-\alpha) = k \tag{3.8}$$

$$c^{\beta_1} c_1 \cdot \beta_1(1-\alpha\beta_1) + c^{\beta_2} c_2 \cdot \beta_2(1-\alpha\beta_2) + c c_3 \cdot (1-\alpha) = 0. \tag{3.9}$$

Solve equations (3.6)-(3.9) to obtain that

$$c = \frac{-f(1)}{1-\alpha} \frac{\beta_1}{\beta_1 - 1} \frac{\beta_2}{\beta_2 - 1} \frac{\eta_1 - 1}{\eta_1} \cdot k \tag{3.10}$$

$$c_1 = c^{-\beta_1} \frac{1}{\eta_1(\alpha\beta_1 - 1)} \frac{\beta_2(\eta_1 - \beta_1)}{(\beta_2-\beta_1)(\beta_1-1)} \cdot k \tag{3.11}$$

$$c_2 = c^{-\beta_2} \frac{1}{\eta_1(\alpha\beta_2 - 1)} \frac{\beta_1(\beta_2 - \eta_1)}{(\beta_2-\beta_1)(\beta_2-1)} \cdot k \tag{3.12}$$

$$c_3 = -\frac{1}{f(1)}. \tag{3.13}$$

We have the following result, whose proof is given in the Appendix.

Proposition 1. *The function v defined by* (3.5) *and* (3.10)-(3.13) *is a solution to the variational inequality* (3.1)-(3.4), *with free boundary $y = g(x) = \log(cx^\alpha)$.*

3.2 The verification theorem

In this section, we prove the following verification theorem.

Theorem 1. *The function v, defined by (3.5) and (3.10)-(3.13), is the value function of the optimization problem. That is,*

$$v(x,y) = V(x,y) = \sup_{X\in\mathcal{A}(x)} \limsup_{T\to\infty} \mathsf{E}\int_0^T e^{-rt}\left(e^{Y_t}H(X_t)\,dt - kdX_t\right).$$

Furthermore, the optimal capital policy is determined by

$$X_t^* \doteq x + \left[\max_{0\le s\le t}\left(g^{-1}(Y_s)-x\right)\right]^+, \quad \textit{or,} \quad X_t^* = \max\left\{x, \max_{0\le s\le t} g^{-1}(Y_s)\right\}; \tag{3.14}$$

here g^{-1} is the inverse of g; that is,

$$g^{-1}(y) = c^{-\frac{1}{\alpha}}\cdot e^{\frac{y}{\alpha}}, \qquad \forall \;\; y\in\mathbb{R}. \tag{3.15}$$

The following lemma is useful to the proof of the theorem.

Lemma 1. *Suppose ξ,η are non-negative random variables and $\beta\in(0,1)$ is a constant. If $\mathsf{E}\xi = +\infty$ but $\mathsf{E}\eta<+\infty$, then*

$$\mathsf{E}\left(\eta^\beta\cdot\xi^{1-\beta}-\xi\right) = -\infty.$$

Proof of Lemma 1. We can write

$$\eta^\beta\cdot\xi^{1-\beta}-\xi = \left(\eta^\beta\cdot\xi^{1-\beta}-\xi\right)\cdot 1_{\{\xi\ge 2\eta\}} + \left(\eta^\beta\cdot\xi^{1-\beta}-\xi\right)\cdot 1_{\{\xi< 2\eta\}} = (\mathrm{I}) + (\mathrm{II}).$$

On set $\{\xi\le 2\eta\}$, we have

$$\eta^\beta\cdot\xi^{1-\beta}-\xi \le 2^{1-\beta}\eta.$$

It follows that $\mathsf{E}(\mathrm{II})<+\infty$, and we only need to show $\mathsf{E}(\mathrm{I})=-\infty$. However, on set $\{\xi\ge 2\eta\}$, we have

$$\eta^\beta\cdot\xi^{1-\beta}-\xi \le (2^{-\beta}-1)\xi.$$

Therefore, it is sufficient to show that

$$\mathsf{E}\left(\xi\cdot 1_{\{\xi\ge 2\eta\}}\right) = +\infty.$$

But this is trivial from the assumptions. □.

Proof of Theorem 1. We first show that v is an upper-bound. Fix an arbitrary capital policy $X\in\mathcal{A}(x)$. We want to show that

$$v(x,y) \ge \limsup_{T\to\infty} \mathsf{E}\int_0^T e^{-rt}\left(e^{Y_t}H(X_t)\,dt - kdX_t\right). \tag{3.16}$$

We consider two separate cases. In the first case, we assume that there is an $T_0\ge 0$ such that $\mathsf{E}X_{T_0}=+\infty$. It is then easy to see that

$$\mathsf{E}\int_0^T e^{-rt}\left(e^{Y_t}H(X_t)\,dt - kdX_t\right) \le X_T^{1-\alpha}\cdot\int_0^T e^{Y_t}\,dt - ke^{-rT}X_T; \qquad \forall\; T\ge T_0.$$

However, Hölder inequality yields that

$$\mathsf{E}\left(\int_0^T e^{Y_t}\,dt\right)^{\frac{1}{\alpha}} \le T^{\frac{1-\alpha}{\alpha}}\mathsf{E}\int_0^T e^{\frac{Y_t}{\alpha}}\,dt < +\infty;$$

here the last inequality follows since $\eta_1 > \frac{1}{\alpha}$, thanks to (2.8). We obtain that

$$\mathsf{E}\int_0^T e^{-rt}\left(e^{Y_t}H(X_t)\,dt - kdX_t\right) = -\infty; \qquad \forall\; T \ge T_0$$

from Lemma 1, which yields the desired inequality (3.16). As for the second case, we assume that $\mathsf{E}X_T < +\infty$ for all T. Applying the Itô formula to the process $\left\{e^{-rt}v(X_t,Y_t);\; t\ge 0\right\}$, we obtain that

$$\begin{aligned} e^{-rT}v(X_T,Y_T) - v(x,y) &= \int_0^T e^{-rt}\left(-rv + \mu v_y + \frac{1}{2}\sigma^2 v_{yy}\right)(X_{t-},Y_{t-})\,dt \\ &+ \int_0^T e^{-rt}v_x(X_{t-},Y_{t-})\,dX_t^c + \int_0^T e^{-rt}\cdot\sigma v_y(X_{t-},Y_{t-})\,dW_t \\ &+ \sum_{0\le t\le T} e^{-rt}\left[v(X_t,Y_t) - v(X_{t-},Y_{t-})\right]; \end{aligned}$$

here X^c is the continuous part of the increasing process X. However,

$$\begin{aligned} \sum_{0\le t\le T} e^{-rt}\left[v(X_t,Y_t) - v(X_{t-},Y_{t-})\right] &= \sum_{0\le t\le T} e^{-rt}\left[v(X_t,Y_t) - v(X_{t-},Y_t) + v(X_{t-},Y_t) - v(X_{t-},Y_{t-})\right] \\ &= \sum_{0\le t\le T} e^{-rt}\left[v(X_t,Y_t) - v(X_{t-},Y_t)\right] + \int_0^T\int_{\mathbb{R}} e^{-rt}\left[v(X_{t-},Y_{t-}+z) - v(X_{t-},Y_{t-})\right]\nu(dz,dt) \\ &+ \int_0^T\int_{\mathbb{R}} e^{-rt}\left[v(X_{t-},Y_{t-}+z) - v(X_{t-},Y_{t-})\right]\cdot(\mu-\nu)(dz,dt), \end{aligned}$$

where μ is the jump measure corresponding to the process $\left\{t\ge 0;\; \sum_{i=1}^{N_t} Z_i\right\}$, and the measure ν is defined as

$$\nu(dz,dt) = \lambda f_Z(z)\,dzdt.$$

Therefore, we have

$$\begin{aligned} e^{-rT}v(X_T,Y_T) - v(x,y) &= \int_0^T e^{-rt}(-rv + \mathcal{L}v)(X_{t-},Y_{t-})\,dt + \int_0^T e^{-rt}v_x(X_{t-},Y_{t-})\,dX_t^c + M_T \\ &\quad + \sum_{0\le t\le T} e^{-rt}\left[v(X_t,Y_t) - v(X_{t-},Y_t)\right] \\ &\le -\int_0^T e^{-rt}e^{Y_{t-}}H(X_{t-})\,dt + \int_0^T e^{-rt}k\,dX_t + M_T; \end{aligned}$$

here $M = (M_t,\mathcal{F}_t)$ is a local martingale where

$$M_T \doteq \int_0^T e^{-rt}\cdot\sigma v_y(X_{t-},Y_{t-})\,dW_t + \int_0^T\int_{\mathbb{R}} e^{-rt}\left[v(X_{t-},Y_{t-}+z) - v(X_{t-},Y_{t-})\right]\cdot(\mu-\nu)(dz,dt).$$

Since X, Y both make at most countably jumps in time interval $[0,T]$, we have

$$\int_0^T e^{-rt} e^{Y_{t-}} H(X_{t-})\, dt = \int_0^T e^{-rt} e^{Y_t} H(X_t)\, dt,$$

which gives

$$\int_0^T e^{-rt} \left(e^{Y_t} H(X_t)\, dt - k\, dX_t \right) \le v(x,y) + M_T; \qquad \forall \;\; T \ge 0.$$

It suffices to show $\mathsf{E} M_T \le 0$. To this end, we show that $\sup_{0\le t\le T} M_t^-$ is integrable for every $T \ge 0$ (hence M is a supermartingale). Indeed, it follows from the preceding inequality that

$$\sup_{0\le t\le T} \left(M_t^- \right) \le v(x,y) + \int_0^T e^{-rt} k\, dX_t \le v(x,y) + kX_T,$$

which is integrable by assumption. Inequality (3.16) holds again in this case.

It remains to show that v is also a lower-bound; indeed, we will show that

$$v(x,y) \le \liminf_{T\to\infty} \mathsf{E} \int_0^T e^{-rt} \left(e^{Y_t} H(X_t^*)\, dt - k dX_t^* \right). \tag{3.17}$$

Applying Itô rule to the process $\left\{ e^{-rt} v(X_t^*, Y_t);\;\; t\ge 0 \right\}$, it is not difficulty to verify that

$$\int_0^T e^{-rt} \left(e^{Y_t} H(X_t^*)\, dt - k\, dX_t^* \right) + e^{-rT} v(X_T^*, Y_T) = v(x,y) + M_T^*; \qquad \forall \;\; T\ge 0, \tag{3.18}$$

for some local martingale M^*. Thus we only need to show that M^* is a submartingale and

$$\lim_{T\to\infty} \mathsf{E} e^{-rT} v(X_T^*, Y_T) = 0. \tag{3.19}$$

Indeed, if this is the case, then

$$v(x,y) \le \mathsf{E} \int_0^T e^{-rt} \left(e^{Y_t} H(X_t^*)\, dt - k\, dX_t^* \right) + \mathsf{E} e^{-rT} v(X_T^*, Y_T)$$

for all T. In particular,

$$\begin{aligned} v(x,y) &\le \liminf_{T\to\infty} \mathsf{E} \int_0^T e^{-rt} \left(e^{Y_t} H(X_t^*)\, dt - k\, dX_t^* \right) + \lim_{T\to\infty} \mathsf{E} e^{-rT} v(X_T^*, Y_T) \\ &= \liminf_{T\to\infty} \mathsf{E} \int_0^T e^{-rt} \left(e^{Y_t} H(X_t^*)\, dt - k\, dX_t^* \right), \end{aligned}$$

which completes the proof.

To this end, we first show that

$$\lim_{t\to\infty} \mathsf{E} \left[e^{-rt} X_t^* \right] = 0 \tag{3.20}$$

Thanks to (3.14)-(3.15), it suffices to establish

$$\lim_{t\to\infty} \mathsf{E} \left[e^{-rt} e^{\max_{0\le s\le t} \frac{Y_s}{\alpha}} \right] = 0;$$

However, observe that for any constant $0 \le \theta < \eta_1$, the process $U = (U_t, \mathcal{F}_t)$ is a martingale; here

$$U_t \doteq e^{\theta Y_t - [f(\theta)+r]t}; \qquad t \ge 0,$$

and $\mathsf{E}U_t \equiv e^{\theta y}$ for all t. We choose and fix an arbitrary $\theta \in (1/\alpha, \eta_1)$ such that $f(\theta) < 0$. The existence of such θ is guaranteed by the assumption (2.8)-(2.9) and the continuity of f. It follows from Doob's inequality that

$$\mathsf{P}\left(\max_{0\le s\le t} \frac{Y_s}{\alpha} \ge u\right) \le \mathsf{P}\left(\max_{0\le s\le t} U_s \ge e^{\alpha\theta u - [f(\theta)+r]^+ t}\right) \le e^{\theta y} \cdot e^{-\alpha\theta u + [f(\theta)+r]^+ t};$$

here $x^+ \doteq \max\{x, 0\}$. Therefore, we have

$$\begin{aligned} \mathsf{E}\left[e^{-rt} e^{\max_{0\le s\le t} \frac{Y_s}{\alpha}}\right] &= e^{-rt} \int_{-\infty}^{\infty} \mathsf{P}\left(\max_{0\le s\le t} \frac{Y_s}{\alpha} \ge u\right) e^u \, du \\ &\le e^{-rt}\left(1 + e^{\theta y} \int_0^{\infty} e^{-\alpha\theta u + [f(\theta)+r]^+ t} e^u \, du\right) \\ &\le e^{-rt} + \frac{e^{\theta y}}{\alpha\theta - 1} e^{\max\{f(\theta), -r\}\cdot t}, \end{aligned}$$

and (3.20) follows. Furthermore, (3.14)-(3.15) imply that

$$e^{\max_{0\le t\le T} \frac{Y_t}{\alpha}} \le C_1 X_T^*, \qquad \forall\, T \ge 0$$

for some positive constant C_1. But inequality $v_x \le k$ and (A.1) imply that, for every $t \ge 0$

$$v(X_t^*, Y_t) \le kX_t^* + v(0, Y_t) = kX_t^* + \left(c^{\beta_1} c_1 + c^{\beta_2} c_2 + cc_3 - k\right) \cdot c^{-\frac{1}{\alpha}} e^{\frac{Y_t}{\alpha}} \le C_2 X_t^*,$$

for some positive constant C_2. (3.19) now follows readily from (3.20) and that $v \ge 0$. We proceed to show that M^* is indeed a submartingale. Note that, since X^* is non-decreasing, we have

$$\begin{aligned} \sup_{0\le t\le T} (M_t^*)^+ &\le \int_0^T e^{-rt} e^{Y_t} H(X_t^*)\, dt + \sup_{0\le t\le T} e^{-rt} v(X_t^*, Y_t) \le \int_0^T e^{-rt} (C_1 X_t^*)^{\alpha} H(X_t^*)\, dt + C_2 X_T^* \\ &= C_1^{\alpha} \int_0^T e^{-rt} X_t^*\, dt + C_2 X_T^* \le \left(r^{-1} C_1^{\alpha} + C_2\right) X_T^*; \qquad \forall\, T \ge 0. \end{aligned}$$

But the proof of (3.20) clearly implies that X_T^* is integrable, whence $\sup_{0\le t\le T} (M_t^*)^+$ is integrable for every $T \ge 0$, which in turn implies that M^* is a submartingle. We complete the proof. □

Remark 5. It follows from the proof that, when $X \equiv X^*$, the lim sup can be replaced by a true lim; that is

$$V(x, y) = \lim_{T\to\infty} \mathsf{E} \int_0^T e^{-rt} \left(e^{Y_t} H(X_t^*)\, dt - k dX_t^*\right)$$

Moreover, we have

$$V(x, y) = \mathsf{E} \int_0^{\infty} e^{-rt} \left(e^{Y_t} H(X_t^*)\, dt - k dX_t^*\right).$$

Indeed, it is not difficult to see that this is a direct consequence of the following two inequalities:

$$\mathsf{E}\int_0^\infty e^{-rt}e^{Y_t}H(X_t^*)\,dt < \infty, \qquad \mathsf{E}\int_0^\infty e^{-rt}dX_t^* < \infty.$$

But the proof of (3.20) implies that $\int_0^\infty e^{-rt}X_t^*\,dt$ is integrable. The first inequality follows readily since $e^{Y_t} \le (C_1X_t^*)^\alpha$, while as for the second inequality, observe that

$$\begin{aligned}\mathsf{E}\int_0^\infty e^{-rt}dX_t^* &= \lim_{T\to\infty}\mathsf{E}\int_{[0,T]} e^{-rt}dX_t^* = \lim_{T\to\infty}\left(\mathsf{E}e^{-rT}X_T^* - x + \mathsf{E}\int_0^T re^{-rt}X_t^*\,dt\right)\\ &= \mathsf{E}\int_0^\infty re^{-rt}X_t^*\,dt - x < \infty.\end{aligned}$$

REMARK 6. The assumption (2.7) assures $V(x,y)$ to be finite for all (x,y). It is not difficult to see that the value function V is infinity if this assumption is violated. Indeed, write the value function as $V_\alpha(x,y) = v_\alpha(x,y)$ to distinguish parameter α. For $x \ge 1$, the value function $V_\alpha(x,y)$ is clearly non-increasing with respect to α. However, as $\alpha \downarrow \beta_1^{-1}$, it is not hard to verify from (3.10)-(3.13) that $c_1 \to \infty$ while $c \to c^*$ for some constant c^*. Hence $V_\alpha(x,y) \to \infty$ at least on the region $\{(x,y);\ x \ge 1\}$, this in turns implies that $V_\alpha \to \infty$ is always true for all (x,y) since $(V_\alpha)_x \le k$ always holds.

Summary

This paper studies an irreversible capacity expansion (incremental investment) problem where the state process is modeled by a double exponential jump diffusion process. Explicit solution is found for the case of Cobb-Douglas production function. For general jump sizes and general production function, one could write out a verification theorem, but it seems very difficult, if not impossible, to explicitly solve the associated variational inequality. Further extension of this work can be made for the case with reversible capacity choices or depreciating capitals.

Appendix. Proof of Proposition 1

Since the constants $(c_1, c_2, c_3; c)$ are all non-negative, thanks to inequalities (2.8) and (2.9), function v is clearly non-negative. Now that we have explicit formula (3.5), its $\mathcal{C}^2$-smoothness can be verified by straightforward computation. Here we only show that v_{yy} is continuous as an example. One can rewrite (3.5) in the following form.

$$\text{(A.1)}\qquad v(x,y) = \begin{cases} c_1x^{1-\alpha\beta_1}e^{\beta_1 y} + c_2x^{1-\alpha\beta_2}e^{\beta_2 y} + c_3x^{1-\alpha}e^y & ; \quad \forall\ x \ge 0,\ y \le \log(cx^\alpha) \\ kx + \left(c^{\beta_1}c_1 + c^{\beta_2}c_2 + cc_3 - k\right)\cdot c^{-\frac{1}{\alpha}}e^{\frac{y}{\alpha}} & ; \quad \forall\ x \ge 0,\ y > \log(cx^\alpha) \end{cases}.$$

It follows that

$$v_{yy}(x,y) = \begin{cases} c_1\beta_1^2x^{1-\alpha\beta_1}e^{\beta_1 y} + c_2\beta_2^2x^{1-\alpha\beta_2}e^{\beta_2 y} + c_3x^{1-\alpha}e^y & ; \quad \forall\ x \ge 0,\ y \le \log(cx^\alpha) \\ \frac{1}{\alpha^2}\left(c^{\beta_1}c_1 + c^{\beta_2}c_2 + cc_3 - k\right)\cdot c^{-\frac{1}{\alpha}}e^{\frac{y}{\alpha}} & ; \quad \forall\ x \ge 0,\ y > \log(cx^\alpha) \end{cases}.$$

In order to show v_{yy} is continuous across the free boundary $y = \log(cx^\alpha)$, it suffices to show that

$$c^{\beta_1}c_1 \cdot \beta_1^2 + c^{\beta_2}c_2 \cdot \beta_2^2 + cc_3 = \frac{1}{\alpha^2}\left(c^{\beta_1}c_1 + c^{\beta_2}c_2 + cc_3 - k\right).$$

However, multiplying (3.9) by α and adding to (3.8), this equality follows readily.

Now we show the inequality (3.3). However, in region $\{(x,y);\ y < \log(cx^\alpha)\}$ we have

$$v_x(x,y) = c_1(1-\alpha\beta_1)x^{-\alpha\beta_1}e^{\beta_1 y} + c_2(1-\alpha\beta_2)x^{-\alpha\beta_2}e^{\beta_2 y} + c_3(1-\alpha)x^{-\alpha}e^y := F\left(\frac{e^y}{cx^\alpha}\right),$$

where

$$F(z) \doteq c^{\beta_1}c_1(1-\alpha\beta_1)z^{\beta_1} + c^{\beta_2}c_2(1-\alpha\beta_2)z^{\beta_2} + cc_3(1-\alpha)z.$$

Therefore it is sufficient to show that $F(z) < k$ for all $0 \le z < 1$. Indeed, F is a strictly concave function, thanks to inequality (2.8) again, with $F(1) = k,\ F(0) = 0$ and

$$F'(1) = c^{\beta_1}c_1 \cdot \beta_1(1-\alpha\beta_1) + c^{\beta_2}c_2 \cdot \beta_2(1-\alpha\beta_2)z^{\beta_2} + cc_3(1-\alpha) = 0$$

from (3.9). Hence $0 \le F(z) < k$ for all $0 \le z < 1$.

It remains to show (3.4). To ease exposition, define $G(x,y) \doteq -rv + \mathcal{L}v + e^y H(x)$. In region $\{(x,y);\ y \ge \log(cx^\alpha)\}$, straightforward computation yields

$$\begin{aligned}
G(x,y) &= e^y x^{1-\alpha} - (r+\lambda)v + \frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial y^2} + \mu\frac{\partial v}{\partial y} + \lambda\int_0^\infty v(x,y+z)\cdot p\eta_1 e^{-\eta_1 z}\,dz \\
&\quad + \lambda\int_{\log(cx^\alpha)-y}^0 v(x,y+z)\cdot q\eta_2 e^{\eta_2 z}\,dz + \lambda\int_{-\infty}^{\log(cx^\alpha)-y} v(x,y+z)\cdot q\eta_2 e^{\eta_2 z}\,dz \\
&= e^y x^{1-\alpha} - rkx + \left(c^{\beta_1}c_1 + c^{\beta_2}c_2 + cc_3 - k\right)\cdot c^{-\frac{1}{\alpha}}e^{\frac{y}{\alpha}} f\left(\frac{1}{\alpha}\right) + \frac{c^{\eta_2}x^{1+\alpha\eta_2}e^{-\eta_2 y}}{1+\alpha\eta_2}\cdot A,
\end{aligned}$$

where constant A is defined as

$$A \doteq \lambda q\eta_2\left(c^{\beta_1}c_1 \cdot \frac{1-\alpha\beta_1}{\eta_2+\beta_1} + c^{\beta_2}c_2 \cdot \frac{1-\alpha\beta_2}{\eta_2+\beta_2} + cc_3 \cdot \frac{1-\alpha}{\eta_2+1}\right) - \lambda kq.$$

We first show that $A < 0$. To this end, observe that equalities (3.8) and (3.11), (3.12) imply

$$\begin{aligned}
A &= \lambda q\eta_2\left[c^{\beta_1}c_1 \cdot \left(\frac{1-\alpha\beta_1}{\eta_2+\beta_1} - \frac{1-\alpha\beta_1}{\eta_2+1}\right) + c^{\beta_2}c_2 \cdot \left(\frac{1-\alpha\beta_2}{\eta_2+\beta_2} - \frac{1-\alpha\beta_2}{\eta_2+1}\right)\right] + \frac{\lambda q\eta_2}{\eta_2+1}k - \lambda kq \\
&= \frac{\lambda q\eta_2}{\eta_2+1}\left[c^{\beta_1}c_1 \cdot \frac{(1-\alpha\beta_1)(1-\beta_1)}{\eta_2+\beta_1} + c^{\beta_2}c_2 \cdot \frac{(1-\alpha\beta_2)(1-\beta_2)}{\eta_2+\beta_2}\right] - \frac{\lambda kq}{\eta_2+1} \\
&= \frac{\lambda kq\eta_2}{\eta_2+1}\left[\frac{\beta_2(\eta_1-\beta_1)}{\eta_1(\beta_2-\beta_1)(\eta_2+\beta_1)} + \frac{\beta_1(\beta_2-\eta_1)}{\eta_1(\beta_2-\beta_1)(\eta_2+\beta_2)} - \frac{1}{\eta_2}\right] \\
&= -\frac{\lambda kq}{\eta_2+1}\cdot\frac{\beta_1\beta_2(\eta_1+\eta_2)}{\eta_1(\eta_2+\beta_1)(\eta_2+\beta_2)}.
\end{aligned}$$

Secondly we show that

$$A = rk - c(1-\alpha) + \sigma^2 \cdot \frac{\beta_1\beta_2}{2\eta_1}k. \tag{A.2}$$

Using the identity

$$\frac{\lambda q\eta_2}{\eta_2+\beta} = f(\beta) + (r+\lambda) - \frac{1}{2}\sigma^2\beta^2 - \mu\beta - \frac{\lambda p\eta_1}{\eta_1-\beta}$$

and $f(\beta_1) = f(\beta_2) = 0$, we have

$$\begin{aligned} A &= f(1)cc_3(1-\alpha) + (r+\lambda)\cdot\left[c^{\beta_1}c_1\cdot(1-\alpha\beta_1) + c^{\beta_2}c_2\cdot(1-\alpha\beta_2) + cc_3\cdot(1-\alpha)\right] \\ &\quad - \frac{1}{2}\sigma^2\cdot\left[c^{\beta_1}c_1\cdot\beta_1^2(1-\alpha\beta_1) + c^{\beta_2}c_2\cdot\beta_2^2(1-\alpha\beta_2) + cc_3\cdot(1-\alpha)\right] \\ &\quad - \mu\cdot\left[c^{\beta_1}c_1\cdot\beta_1(1-\alpha\beta_1) + c^{\beta_2}c_2\cdot\beta_2(1-\alpha\beta_2) + cc_3\cdot(1-\alpha)\right] \\ &\quad - \lambda p\eta_1\cdot\left[c^{\beta_1}c_1\cdot\frac{1-\alpha\beta_1}{\eta_1-\beta_1} + c^{\beta_2}c_2\cdot\frac{1-\alpha\beta_2}{\eta_1-\beta_2} + cc_3\cdot\frac{1-\alpha}{\eta_1-1}\right] - \lambda kp \\ &= -c(1-\alpha) + rk - \frac{1}{2}\sigma^2\cdot\left[c^{\beta_1}c_1\cdot\beta_1^2(1-\alpha\beta_1) + c^{\beta_2}c_2\cdot\beta_2^2(1-\alpha\beta_2) + cc_3\cdot(1-\alpha)\right], \end{aligned}$$

thanks to (3.6)-(3.9) and $p+q=1$. It remains to show that

$$c^{\beta_1}c_1\cdot\beta_1^2(1-\alpha\beta_1) + c^{\beta_2}c_2\cdot\beta_2^2(1-\alpha\beta_2) + cc_3\cdot(1-\alpha) = -\frac{\beta_1\beta_2}{\eta_1}k.$$

But this equality can be verified using (3.10)-(3.12) and direct calculation. We omit the details.

Finally we show that $G(x,y) < 0$ in the region $\{(x,y);\ y > \log(cx^\alpha)\}$. Fix y, and we consider G as a function of x on interval $\left[0, g^{-1}(y)\right]$. Since $A < 0$, it is easy to see that G is indeed strictly concave. However, note that $G(x,y) = 0$ at $x = g^{-1}(y)$ by the construction of function v, as well as its $\mathcal{C}^2$-smoothness. It is sufficient to show that $G_x(x,y)|_{x=g^{-1}(y)} > 0$. But by direct computation we have

$$G_x(x,y)|_{x=g^{-1}(y)} = c(1-\alpha) - rk + A = \sigma^2\frac{\beta_1\beta_2}{2\eta_1}k,$$

thanks to equality (A.2). This completes the proof. □

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