# Importance Sampling for Weighted-Serve-the-Longest-Queue

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February 29, 2008

#### Abstract

This paper considers buffer overflow probabilities for stable queueing systems with one server and different classes of arrivals. The service priority is given to the class of customers whose current weighted queue size is the largest (weighted-serve-the-longest-queue policy). We explicitly identify the exponential decay rate for the rare-event probabilities of interest, and construct asymptotically optimal importance sampling schemes for simulation.

### 1 Introduction

It is common for communication and manufacturing systems to have different classes of customer arrivals. A popular service discipline in such situations is the serve-the-longest-queue policy or its natural generalization, the *weighted-serve-the-longest-queue* (WSLQ) policy. Under the WSLQ policy, each class is given a prefixed weight, and the class of customers with the largest weighted queue length is assigned the service priority.

This paper studies efficient rare-event simulation techniques for stable WSLQ systems with one server and multiple classes of arrivals. Service

<sup>\*</sup>Research of this author supported in part by the National Science Foundation (NSF-DMS-0706003) and the Army Research Office (W911NF-05-1-0289).

<sup>&</sup>lt;sup>†</sup>Research of this author supported in part by the National Science Foundation (NSF-DMS-0706003)

 $<sup>^{\</sup>ddagger} Research of this author supported in part by the National Science Foundation (NSF-DMS-0706003).$ 

policies such as WSLQ fall into the general category of systems with *discontinuous dynamics* and are difficult to analyze [1, 4, 8, 9, 10, 12, 11]. A general sample path large deviations result for WSLQ systems was only recently established [5].

The present paper concentrates on a class of individual buffer overflow probabilities and resolves two questions: (1) the explicit identification of the exponential decay rate of such overflow probabilities; (2) the construction of asymptotically optimal importance sampling schemes for fast simulation of such rare events. These two questions turn out to be intimately connected in the following sense. To build efficient importance sampling schemes, we use the game/subsolution approach [7] and construct suitable subsolutions to the associated Isaacs equations. Not only do these subsolutions lead to asymptotically optimal importance sampling schemes, they also play an essential role in explicitly identifying the exponential decay rate of the overflow probabilities via a verification argument.

The serve-the-longest policy has been studied in the context of wireless communications, see for example [13, 2] and their references. There are differences between the stochastic processes used in the wireless literature and the processes used in our work, the most significant of which is that the dynamics in the wireless context are not Markovian, but rather modulated by an exogenous Markov process. However the methods used to deal with the discontinuous interfaces in our paper can be combined with those presented in [7] for Markov modulated rates to analyze such models.

The paper is organized as follows. In Sections 2 and 3 the model dynamics and problem formulation are described. Section 4 presents the large deviations result regarding the overflow probabilities. A brief review of importance sampling is conducted in section 5, and the associated Isaacs equation is formally derived in Section 6. A classical subsolution to the Isaacs equation and the corresponding importance sampling scheme are constructed in Section 7. In Section 8 we derive the exponential decay rate of the overflow probabilities and establish the asymptotic optimality of the said importance sampling scheme. Numerical results are presented in Section 9. Many technical proofs are collected in an appendix.

# 2 Problem Formulation

The system model consists of a single server and d classes of customers. Customers of class i, buffered at queue i, arrive according to a Poisson process with rate  $\lambda_i$ . The service time for a class i customer is exponentially distributed with rate  $\mu_i$ . Upon the completion of service, the customer leaves the system. We will assume that the system satisfies the stability condition



Figure 1: WSLQ system

The service policy is determined according to the following WSLQ discipline. Let  $c_i$  be the weight associated with class i. If the size of queue i is  $q_i$ , then the weighted length for queue i is defined as  $c_i q_i$ . Under the WSLQ policy, the queue with the maximal weighted length is assigned the service priority. When there are multiple queues with the maximal weighted length, the assignment of priority among these queues is non-essential and can be arbitrary. We adopt the convention that when there are ties, the priority will be given to the queue with the largest index.

Denote the system state at time t by  $Q(t) \doteq (Q_1(t), \ldots, Q_d(t))$ , where  $Q_i$  is the size of queue i. The process Q is a continuous time Markov jump process defined on some probability space, say  $(\Omega, \mathcal{F}, \mathbb{P})$ . The quantities of interest are the buffer overflow probabilities

$$p_n \doteq \mathbb{P}\{\text{the process } Q \text{ exits domain } nD \text{ before returning}$$
to the origin, starting from the origin}, (2.2)

where n is a large integer and D is the domain

$$D \doteq \{x = (x_1, \dots, x_d) \in \mathbb{R}^d_+ : c_i x_i \le 1 \text{ for all } i\}.$$
 (2.3)

We also define the vector

$$\bar{c} = (1/c_1, 1/c_2, \dots, 1/c_d),$$
(2.4)

which is a corner of D, and denote the boundary of D in  $\mathbb{R}^d_+$  by

$$\partial \doteq \partial D = \{x \in \mathbb{R}^d_+ : x \in D, c_i x_i = 1 \text{ for some } i\}.$$



Figure 2: System dynamics with d = 2.

See Figure 2 for illustration of the case of dimension d = 2.

Throughout the analysis, it is often convenient to consider the scaled process

$$X^n(t) \doteq Q(nt)/n.$$

Note that we can now rewrite the overflow probability as

 $p_n = \mathbb{P}\{X^n \text{ exits domain } D \text{ before returning to the origin,} starting at the origin}\}.$ 

# 3 System Dynamics

The collection of possible jumps for the process Q is denoted by

$$\mathbb{V} \doteq \{\pm e_j : j = 1, 2, \dots, d\},\$$

where  $e_i$  is the canonical unit vector with the *i*-th component 1 and other components 0. For  $v \in \mathbb{V}$ , let r(x, v) be the jump intensity of the process Qfrom state x to state x + v. For  $x = (x_1, \ldots, x_d) \in \mathbb{R}^d_+$  and  $x \neq 0$ , let  $\pi(x)$ denote the indices of queues that have the maximal weighted length, that is,

$$\pi(x) \doteq \left\{ 1 \le i \le d : c_i x_i = \max_j c_j x_j \right\}.$$

Then

$$r(x;v) = \begin{cases} \lambda_i & \text{if } v = e_i \text{ and } i = 1, \dots, d, \\ \mu_i & \text{if } v = -e_i \text{ where } i = \max \pi(x), \\ 0 & \text{otherwise.} \end{cases}$$

For x = 0, there is no service and the jump intensities are

$$r(0; v) = \begin{cases} \lambda_i & \text{if } v = e_i \text{ and } i = 1, \dots, d_i \\ 0 & \text{otherwise.} \end{cases}$$

We also set

$$\pi(0) \doteq \{0, 1, 2, \dots, d\}$$

The dynamics of the process Q can be described by a stochastic transition kernel, say  $\Theta[dt, v|x]$ . To be more precise, define the total intensity by

$$R(x) \doteq \sum_{v \in \mathbb{V}} r(x, v), \qquad (3.1)$$

and let  $T_1, T_2, \ldots$  be the random times at which the process Q jumps with convention  $T_0 = 0$ . Then for any  $x \in \mathbb{R}^d_+$  and  $n \in \mathbb{N}$  such that  $nx \in \mathbb{Z}^d_+$ ,

$$\Theta[dt, v|x] \doteq \mathbb{P}\{T_{j+1} - T_j \in dt, Q(T_{j+1}) = nx + v|Q(T_j) = nx\}$$
  
=  $r(x, v)e^{-R(x)t}dt.$  (3.2)

# 4 Large Deviations

One goal of this paper is to explicitly identify the exponential decay rate of  $p_n$ . Our approach is to reduce it to a calculus of variation problem using the sample path large deviations principle established in [5]. In order to explicitly solve the calculus of variation problem, we will use a verification argument based on essentially the same subsolution that we will use to build asymptotically optimal importance sampling schemes.

To ease notation, we will also denote by A the collection of non-empty subsets of  $\{1, 2, \ldots, d\}$ . That is

$$\mathbb{A} = \{A : A \subset \{1, 2..., d\}, A \text{ is non-empty}\}.$$
(4.1)

#### 4.1 The rate functions

For each i = 1, ..., d, let  $H^{(i)}$  be the convex function given by

$$H^{(i)}(\alpha) \doteq \mu_i(e^{-\alpha_i} - 1) + \sum_{j=1}^d \lambda_j(e^{\alpha_j} - 1),$$

for all  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$ . We also define for i = 0 and  $\alpha \in \mathbb{R}^d$ 

$$H^{(0)}(\alpha) \doteq \sum_{j=1}^{d} \lambda_j (e^{\alpha_j} - 1)$$

For each non-empty subset  $A \subset \{1, \ldots, d\}$  or  $A = \{0, 1, \ldots, d\}$ , let  $L^A$  be the Legendre transform of  $\max_{i \in A} H^{(i)}$ , that is,

$$L^{A}(\beta) \doteq \sup_{\alpha \in \mathbb{R}^{d}} \left[ \langle \alpha, \beta \rangle - \max_{i \in A} H^{(i)}(\alpha) \right]$$

for each  $\beta \in \mathbb{R}^d$ . Clearly  $L^A$  is convex and, since  $H^{(i)}(0) = 0$ , it is also non-negative. When A is a singleton  $\{i\}$ , we simply write  $L^A$  as  $L^{(i)}$ . A useful representation of  $L^A$  [4, Corollary D.4.3] is

$$L^{A}(\beta) = \inf \sum_{i \in A} \rho^{(i)} L^{(i)}(\beta^{(i)}), \qquad (4.2)$$

where the infimum is taken over all  $\{(\rho^{(i)}, \beta^{(i)}) : i \in A\}$  such that

$$\rho^{(i)} \ge 0, \quad \sum_{i \in A} \rho^{(i)} = 1, \quad \sum_{i \in A} \rho^{(i)} \beta^{(i)} = \beta.$$
(4.3)

Furthermore, for  $x \in \mathbb{R}^d_+$  let  $L(x,\beta) \doteq L^{\pi(x)}(\beta)$ . It turns out that L is the local rate function associated with the scaled processes  $\{X^n\}$ , see [5].

#### 4.2 The exponential decay rate

In this section we first identify a collection of important roots associated with the Hamiltonians  $H^{(i)}$ , i = 1, ..., d. The motivation for such roots will be discussed in Remark 4.4. These roots will be used to construct subsolutions and identify the large deviation rate for  $p_n$ .

For each  $A \in \mathbb{A}$ , thanks to the stability condition (2.1), there exists a unique constant, say  $z^A$ , such that

$$0 < z^A < \min_{i \in A} \mu_i, \qquad \sum_{j \in A} \frac{\lambda_j}{\mu_j - z^A} = 1.$$
 (4.4)

Define the vector  $\alpha^A \in \mathbb{R}^d$  by

$$\alpha_i^A \doteq \begin{cases} \log\left(1 - \frac{z^A}{\mu_i}\right) & \text{if } i \in A, \\ 0 & \text{otherwise.} \end{cases}$$
(4.5)

We have the following result, whose proof is elementary and thus omitted.

**Lemma 4.1.** For each  $A \in \mathbb{A}$ ,  $\alpha^A$  is the unique vector that satisfies

1.  $\alpha_i^A = 0$  for  $i \notin A$ , 2.  $\alpha_i^A < 0$  for  $i \in A$ , 3.  $H^{(i)}(-\alpha^A) = 0$  for all  $i \in A$ .

4. 
$$H^{(i)}(-\alpha^A) > 0$$
 for all  $i \notin A$ .

Recall the definition of  $\bar{c}$  as in (2.4) and define

$$\gamma \doteq \min\left\{ \langle -\alpha^A, \bar{c} \rangle : A \in \mathbb{A} \right\}.$$
(4.6)

The following result identifies  $\gamma$  as the value of the calculus of variation problem that is associated with the large deviations of  $\{p_n\}$ . The proof is technical and deferred to the appendix.

**Theorem 4.2.** We have the representation

$$\gamma = \inf \int_0^\tau L(\phi(t), \dot{\phi}(t)) \, dt,$$

where the infimum is taken over all absolutely continuous functions  $\phi : \mathbb{R}_+ \to \mathbb{R}^d_+$  such that  $\phi(0) = 0$  and  $\tau \doteq \inf\{t \ge 0 : \phi(t) \in \partial\} < \infty$ .

**Remark 4.3.** It will be shown later on that  $\gamma$  is the exponential decay rate of  $\{p_n\}$  [Theorem 8.3], that is,

$$\gamma = -\lim_n \frac{1}{n} \log p_n.$$

**Remark 4.4.** Large deviations rate functions are closely related to control problems and thus are also related to the solutions of appropriate Hamiltonian-Jacobi-Bellman (HJB) equations [4]. Due to the homogeneity of the state dynamics, it is natural to conjecture that the large deviation optimal trajectory leaving the domain will be a straight line, and the solution to the HJB equation is piecewise affine. If the optimal trajectory leaves the domain through interface  $\{x : \pi(x) = A\}$ , then it corresponds to an affine piece whose gradient, say v, should satisfy the HJB equation  $H^{(i)}(v) = 0$  for all  $i \in A$ . Furthermore, v should satisfy  $\langle v, e_i \rangle = 0$  for all  $i \notin A$  since the value function to the corresponding control problem will take the same value 0 on the boundary  $\{x : \pi(x) = A\} \cap \partial$ . Solving these equations about v yields  $v = -\alpha^A$ .

# 5 Importance Sampling

We are interested in efficient importance sampling schemes for estimating  $p_n$  when n is large. Importance sampling simulates the system under a different probability distribution, i.e., change of measure.

#### 5.1 Asymptotic optimality

Denote by  $A_n$  the event of buffer overflow, and rewrite  $p_n = \mathbb{P}(A_n)$ . An importance sampling scheme generates samples from a new probability measure, say  $\mathbb{Q}_n$ , such that  $\mathbb{P} \ll \mathbb{Q}_n$ . The estimator is then given by the average of independent replications of

$$\hat{p}_n \doteq 1_{A_n} \frac{d\mathbb{P}}{d\mathbb{Q}_n},$$

where  $d\mathbb{P}/d\mathbb{Q}_n$  is the Radon-Nikodym derivative or likelihood ratio. Clearly  $\hat{p}_n$  is unbiased for any such  $\mathbb{Q}_n$ .

The goal of importance sampling is to choose  $\mathbb{Q}_n$  to minimize the variance, or the second moment of  $\hat{p}_n$ . An obvious lower bound follows from Jensen's inequality and the large deviations properties of  $p_n$  [Theorem 4.2 and Remark 4.3],

$$\liminf_{n} \frac{1}{n} \log E^{\mathbb{Q}_n} \left[ \hat{p}_n^2 \right] \ge \liminf_{n} \frac{2}{n} \log E^{\mathbb{Q}_n} \left[ \hat{p}_n \right] = \liminf_{n} \frac{2}{n} \log p_n = -2\gamma.$$

An importance sampling scheme, or the change of measure  $\mathbb{Q}_n$ , is said to be *asymptotically optimal* if this lower bound is achieved, i.e., if

$$\limsup_{n} \frac{1}{n} \log E^{\mathbb{Q}_n} \left[ \hat{p}_n^2 \right] \le -2\gamma.$$

For future analysis, it is worthwhile to note that the second moment equals

$$E^{\mathbb{Q}_n}\left[\hat{p}_n^2\right] = E^{\mathbb{P}}\left[\hat{p}_n\right].$$
(5.1)

#### 5.2 State-dependent importance sampling schemes

In this paper we will consider state-dependent changes of measure for importance sampling. Such changes of measure will be described by stochastic transition kernels, say  $\overline{\Theta}^n[dt, v|x]$ , on  $\mathbb{R}_+ \times \mathbb{V}$  given  $\mathbb{R}^d_+$ . Thus in contrast to (3.2), the dynamics of the state process Q are now determined by

$$\mathbb{Q}_n \{ T_{j+1} - T_j \in dt, \ Q(T_{j+1}) = nx + v | Q(T_j) = nx \} = \bar{\Theta}^n [dt, v | x]$$

for all  $j \ge 0$ , where  $\{T_1, T_2, \ldots\}$  are the jump times for the process Q with convention  $T_0 = 0$ .

In order to explicitly identify the importance sampling estimator, let  $s_j \doteq T_j - T_{j-1}$  and  $v_j \doteq Q(T_j) - Q(T_{j-1})$ . Define the hitting times

$$N^{n} \doteq \inf \left\{ k \ge 1 : Q(T_{k}) \notin nD \right\}, \tag{5.2}$$

$$N^0 \doteq \inf \{k \ge 1 : Q(T_k) = 0\}.$$
 (5.3)

Recall the original stochastic transition kernel  $\Theta$  as in (3.2). Then a single sample of the the importance sampling estimator is

$$\hat{p}_n \doteq \mathbb{1}_{\{N^n < N^0\}} \prod_{j=1}^{N^n} \frac{\Theta\left[ds_j, v_j | Q(T_{j-1})/n\right]}{\bar{\Theta}^n \left[ds_j, v_j | Q(T_{j-1})/n\right]}.$$
(5.4)

In practice we simulate a number of iid copies of  $\hat{p}_n$ , and use the sample mean to estimate  $p_n$ .

### 6 The Isaacs Equation

The connection between importance sampling and differential games/Isaacs equations was explored in [7]. Since we are now dealing with continuous time processes as opposed to the discrete time setup in [7], we will formally derive the associated Isaacs equation in some detail. While the derivation is formal, the analysis of importance sampling schemes that are based on subsolutions to the equation will be rigorous.

Let  $\mathcal{R}$  be the collection of functions mapping  $\mathbb{V}$  to  $[0, \infty)$  that are not identically zero. Each element in  $\mathcal{R}$  represents a set of jump intensities. We will restrict our attention to those alternative stochastic transition kernels that take the form

$$\bar{\Theta}^n[dt,v|x] = \bar{r}(x,v)e^{-\bar{R}(x)t}\,dt, \quad \bar{r}(x,\cdot) \in \mathcal{R}, \ \bar{R}(x) = \sum_{v \in \mathbb{V}} \bar{r}(x,v). \tag{6.1}$$

Consider the stochastic control problem of minimizing the second moment of  $\hat{p}_n$  [see equation (5.1)] among such transition kernels, and define the value function

$$V_n(x) \doteq \inf_{\bar{\Theta}^n} E^{\mathbb{P}}[\hat{p}_n | Q(0) = nx].$$
(6.2)

Clearly  $V_n(x)$  satisfies the boundary condition  $V_n(x) = 1$  for all  $x \notin D$ .

Note that for  $\overline{\Theta}^n$  defined as in (6.1), one can rewrite the likelihood ratio term in the importance sampling estimator  $\hat{p}_n$  (5.4) as

$$\exp\left\{\int_0^{T_{N^n}} \left[\bar{R}(Q(s)/n) - R(Q(s)/n)\right] ds + \sum_{j=1}^{N^n} \log \frac{r(Q(T_{j-1})/n, v_j)}{\bar{r}(Q(T_{j-1})/n, v_j)}\right\}.$$

Therefore, by the dynamic programming principle, the value function  $V_n$  should satisfy the dynamic programming equation (DPE)

$$V_n(x) = \inf_{\bar{r} \in \mathcal{R}} \int_0^\infty \sum_{v \in \mathbb{V}} e^{[\bar{R} - R(x)]t} \cdot \frac{r(x,v)}{\bar{r}(v)} \cdot V_n\left(x + \frac{v}{n}\right) \cdot e^{-R(x)t} r(x,v) dt.$$

Here instead of the fixed state dependent rates we minimize over  $\bar{r}(v)$  and let  $\bar{R} = \sum_{v \in \mathbb{V}} \bar{r}(v)$ . In order to write down a limit partial differential equation, we introduce the following result, which is indeed a special case of the relative entropy representation for exponential integrals [4]. The proof is deferred to the appendix. Note that the function  $\ell : \mathbb{R} \to \mathbb{R}_+ \cup \{\infty\}$  is defined as

$$\ell(x) \doteq \begin{cases} x \log x - x + 1, & \text{if } x \ge 0, \\ \infty & \text{otherwise} \end{cases}$$

with convention  $0\ell(0/0) = 0$  and  $0\ell(x/0) = \infty$  if  $x \neq 0$ .

**Lemma 6.1.** For any  $\theta \in \mathbb{R}$  let

$$s[\theta] \doteq \sum_{v \in \mathbb{V}} \theta(v).$$

Then for any  $\theta \in \mathbb{R}$ , any constant c and function  $h : \mathbb{V} \to \mathbb{R}$ 

$$-\log \int_0^\infty \sum_{v \in \mathbb{V}} e^{-[ct+h(v)]} e^{-s[\theta]t} \theta(v) dt$$
$$= \inf_{\hat{\theta} \in \mathcal{R}} \frac{1}{s[\hat{\theta}]} \left[ c + \sum_{v \in \mathbb{V}} h(v) \hat{\theta}(v) + \sum_{v \in \mathbb{V}} \theta(v) \ell\left(\frac{\hat{\theta}(v)}{\theta(v)}\right) \right].$$

We combine Lemma 6.1 with the DPE of  $V_n$  and a transform motivated by the large deviation scaling

$$W_n(x) \doteq -\frac{1}{n} \log V_n(x)$$

to obtain

$$nW_n(x) = \sup_{\bar{r}\in\mathcal{R}} \inf_{\hat{r}\in\mathcal{R}} \frac{1}{s[\hat{r}]} \left[ R(x) - \bar{R} + \sum_{v\in\mathbb{V}} \hat{r}(v) \log \frac{\bar{r}(v)}{r(x,v)} + \sum_{v\in\mathbb{V}} nW_n \left(x + \frac{v}{n}\right) \hat{r}(v) + \sum_{v\in\mathbb{V}} r(x,v)\ell\left(\frac{\hat{r}(v)}{r(x,v)}\right) \right].$$

To formally obtain a limit PDE, we assume that

$$W_n(x) \to W(x), \quad n \left[ W_n(x+v/n) - W_n(x) \right] \to \langle DW(x), v \rangle.$$
 (6.3)

Observing that

$$R(x) - \bar{R} + \sum_{v \in \mathbb{V}} \hat{r}(v) \log \frac{\bar{r}(v)}{r(x,v)} = \sum_{v \in \mathbb{V}} r(x,v) \ell\left(\frac{\hat{r}(v)}{r(x,v)}\right) - \sum_{v \in \mathbb{V}} \bar{r}(v) \ell\left(\frac{\hat{r}(v)}{\bar{r}(v)}\right),$$

the limit function W satisfies the equation

$$0 = \sup_{\bar{r} \in \mathcal{R}} \inf_{\hat{r} \in \mathcal{R}} \frac{1}{s[\hat{r}]} \left[ \sum_{v \in \mathbb{V}} \langle DW(x), v \rangle \hat{r}(v) + 2 \sum_{v \in \mathbb{V}} r(x, v) \ell\left(\frac{\hat{r}(v)}{r(x, v)}\right) - \sum_{v \in \mathbb{V}} \bar{r}(v) \ell\left(\frac{\hat{r}(v)}{\bar{r}(v)}\right) \right].$$

$$(6.4)$$

Define for every  $x \in \mathbb{R}_+$  and  $\alpha \in \mathbb{R}^d$ 

$$\mathbb{H}(x,\alpha) = \sup_{\bar{r}\in\mathcal{R}} \inf_{\hat{r}\in\mathcal{R}} \left[ \sum_{v\in\mathbb{V}} \hat{r}(v) \langle v,\alpha \rangle + 2\sum_{v\in\mathbb{V}} r(x,v) \ell\left(\frac{\hat{r}(v)}{r(x,v)}\right) - \sum_{v\in\mathbb{V}} \bar{r}(v) \ell\left(\frac{\hat{r}(v)}{\bar{r}(v)}\right) \right].$$
(6.5)

It is not difficult to argue by the existence of saddle points for  $\mathbb{H}$  [Proposition 6.2] that the equation (6.4) for W can be reduced to

$$\mathbb{H}(x, DW(x)) = 0.$$

This is the Isaacs equation associated with a differential game of two players. It should be mentioned that the  $\bar{r}$ -player chooses the change of measure used in the importance sampling scheme and the  $\hat{r}$ -player is an artificial player introduced by the representation formula.

**Proposition 6.2.** For any  $x \in \mathbb{R}^d_+ \setminus \{0\}$  and  $\alpha \in \mathbb{R}^d$ ,

- 1.  $\mathbb{H}(x, \alpha) = \mathbb{H}_j(\alpha) \doteq -2H^{(j)}(-\alpha/2), \text{ where } j = \max\{i : i \in \pi(x)\}.$
- 2. The saddle point is given by

$$\bar{r}^*(x,\alpha)[v] = \hat{r}^*(x,\alpha)[v] = r(x,v)e^{-\langle \alpha,v\rangle/2}.$$

**Remark 6.3.** The Isaacs equation  $\mathbb{H}(x, DW(x)) = 0$  should not be interpreted literally at the points of discontinuities, that is, points x such that  $|\pi(x)| > 1$ . This is easy to understand from the above formal analysis since the second part of the approximation (6.3) will not hold at such x. Instead, a natural interpretation at such points is [3]

$$\lim_{\varepsilon \downarrow 0} \inf_{\{y \in \mathbb{R}^d_+ : \|y - x\| \le \varepsilon\}} \mathbb{H}(y, DW(x)) = 0,$$

which amounts to

$$\min_{j\in\pi(x)}\mathbb{H}_j(DW(x)) = 0.$$
(6.6)

The intuition for this interpretation is that the behavior of the controlled state process on a discontinuity interface is the asymptotic characterization for the collective behavior of the process around the neighborhood of x. This observation is also essential in the construction of classical subsolutions to the Isaacs equation [that is, replace  $\mathbb{H}(x, DW) = 0$  by  $\mathbb{H}(x, DW) \ge 0$ ], since, owing to the infimum operation in (6.6), the subsolution property at x will imply that it holds, at least approximately, for all nearby points of x in the prelimit. See Lemma 7.3 for the precise statement.

# 7 Classical Subsolutions

A continuously differentiable function  $\overline{W}: \mathbb{R}^d_+ \to \mathbb{R}$  is a *classical subsolution* to the Isaacs equation if

- 1.  $\mathbb{H}(x, D\bar{W}(x)) \ge 0$  for all  $x \in D$
- 2.  $\overline{W}(x) \leq 0$  for all  $x \notin D$ .

Classical subsolutions are the means by which the explicit representation for  $\gamma$  in Theorem 4.2 is shown and asymptotically efficient importance sampling schemes are built. We will construct the subsolutions by mollifying piecewise affine subsolutions as in [7]. In order to achieve asymptotic optimality it is necessary that the value of the subsolutions at the origin is maximal, i.e.,  $\bar{W}(0) = 2\gamma$ .

#### 7.1 Piecewise affine subsolutions

Recall that A is the collection of all non-empty subsets of  $\{1, 2, \ldots, d\}$ , the definition of the roots  $\{\alpha^A\}$  in (4.4) and (4.5), and the definition of  $\gamma$  in (4.6). For each  $A \in \mathbb{A}$  define

$$\bar{\alpha}^A = \frac{\gamma}{\langle -\alpha^A, \bar{c} \rangle} \cdot \alpha^A. \tag{7.1}$$

Note that  $\gamma \leq \langle -\alpha^A, \bar{c} \rangle$  for all  $A \in \mathbb{A}$ . Thanks to Lemma 4.1, the convexity of  $H^{(i)}$ , and that  $H^{(i)}(0) = 0$ , we have

Lemma 7.1. For every  $A \in \mathbb{A}$ ,

1.  $\bar{\alpha}_i^A = 0 \text{ for } i \notin A,$ 2.  $\bar{\alpha}_i^A < 0 \text{ for } i \in A,$ 3.  $H^{(i)}(-\bar{\alpha}^A) \leq 0 \text{ for all } i \in A,$ 4.  $\langle -\bar{\alpha}^A, \bar{c} \rangle = \gamma.$ 

Also define a sequence of positive constants  $\{C_1, C_2, \ldots, C_d\}$  by

$$C_1 = 0, \ C_{k+1} = a_k^* \cdot (1 + C_k),$$
 (7.2)

where

$$a_{k}^{*} = 1 \vee \max\left\{\frac{\bar{\alpha}_{i}^{B}}{\bar{\alpha}_{i}^{A}} : i \in A \subset B, |A| = k, |B| = k + 1\right\}$$
(7.3)

for  $1 \le k \le d-1$ . Fix an arbitrary  $\delta > 0$ . Define the affine function

$$W_A^{\delta}(x) \doteq \langle 2\bar{\alpha}^A, x \rangle + 2\gamma - C_{|A|}\delta.$$
(7.4)

for each  $A \in \mathbb{A}$ . Their pointwise minimum is denoted by

$$W^{\delta}(x) \doteq \min\{W^{\delta}_{A}(x) : A \in \mathbb{A}\}.$$
(7.5)

Note that  $W^{\delta}$  is a continuous piecewise affine function.

**Lemma 7.2.** The function  $W^{\delta}$  satisfies  $W^{\delta}(x) \leq 0$  for all  $x \notin D$ . Furthermore, given any  $x \in \mathbb{R}^d_+ \setminus \{0\}$ , if  $W^{\delta}_A(x) - W^{\delta}(x) < \delta$  then  $\pi(x) \subset A$ .

The proof of Lemma 7.2 is deferred to the appendix. This lemma implies that  $W^{\delta}$  is a piecewise affine subsolution, i.e.,  $\mathbb{H}(x, DW^{\delta}(x)) \geq 0$  at all those x where  $DW^{\delta}(x)$  is well defined. Indeed, for any such x, there is a unique  $A^* \in \mathbb{A}$  that attains the minimum in (7.5) and whence  $DW^{\delta}(x) = 2\bar{\alpha}^{A^*}$ . Let  $j = \max\{i : i \in \pi(x)\}$ . It follows from Lemma 7.2 that  $j \in \pi(x) \subset A^*$ . It is now immediate from Lemma 7.2 and Proposition 6.2 that

$$\mathbb{H}(x, DW^{\delta}(x)) = \mathbb{H}_{j}(2\bar{\alpha}^{A^{*}}) = -2H^{(j)}(-\bar{\alpha}^{A^{*}}) \ge 0.$$

#### 7.2 Mollification

The construction of asymptotically optimal importance sampling schemes requires smooth subsolutions with bounded second derivatives. To this end, we consider a mollified version of  $W^{\delta}$  using *exponential weighting* [7, 6]. Let  $\varepsilon$  be an arbitrary small positive number, and define

$$W^{\varepsilon,\delta}(x) \doteq -\varepsilon \log \sum_{A \in \mathbb{A}} \exp\left\{-\frac{1}{\varepsilon}W_A^{\delta}(x)\right\}.$$

The function  $W^{\varepsilon,\delta}$  is smooth with

$$DW^{\varepsilon,\delta}(x) = \sum_{A \in \mathbb{A}} 2\rho_A^{\varepsilon,\delta}(x)\bar{\alpha}^A, \quad \rho_A^{\varepsilon,\delta}(x) \doteq \frac{\exp\left\{-W_A^{\delta}(x)/\varepsilon\right\}}{\sum_{B \in \mathbb{A}} \exp\left\{-W_B^{\delta}(x)/\varepsilon\right\}}.$$

Furthermore, for every x,

$$-\varepsilon \log |\mathbb{A}| \le W^{\varepsilon,\delta}(x) - W^{\delta}(x) \le 0.$$
(7.6)

The following Lemma 7.3 shows that  $W^{\varepsilon,\delta}$  is approximately a subsolution.

**Lemma 7.3.** The function  $W^{\varepsilon,\delta}$  satisfies  $W^{\varepsilon,\delta}(x) \leq 0$  for all  $x \notin D$ . Furthermore, for every  $x \in \mathbb{R}^d_+ \setminus \{0\}$  and  $j \in \pi(x)$ ,

$$\mathbb{H}_j(DW^{\varepsilon,\delta}(x)) \ge \sum_{A \in \mathbb{A}} \rho_A^{\varepsilon,\delta}(x) \cdot \mathbb{H}_j(2\bar{\alpha}^A) \ge -Ke^{-\delta/\varepsilon},$$

where K is a positive constant that only depends on the system parameters. In particular,

$$\mathbb{H}(x, DW^{\varepsilon,\delta}(x)) \ge -Ke^{-\delta/\varepsilon}.$$

**Proof.** We first note that  $W^{\varepsilon,\delta}(x) \leq 0$  for all  $x \notin D$  is trivial from Lemma 7.2 and inequality (7.6). Now let  $x \in \mathbb{R}^d_+ \setminus \{0\}$  and  $j \in \pi(x)$ . Thanks to Proposition 6.2 and the concavity of  $\mathbb{H}_j$ ,

$$\mathbb{H}_j(DW^{\varepsilon,\delta}(x)) \ge \sum_{A \in \mathbb{A}} \rho_A^{\varepsilon,\delta}(x) \cdot \mathbb{H}_j(2\bar{\alpha}^A).$$

For all those  $A \in \mathbb{A}$  such that  $W_A^{\delta}(x) - W^{\delta}(x) < \delta$ , Lemma 7.2 implies  $\pi(x) \subset A$ , in particular  $j \in A$ , and therefore  $\mathbb{H}_j(2\bar{\alpha}^A) = -2H^{(j)}(-\bar{\alpha}^A) \ge 0$  by Lemma 7.1. On the other hand, for those  $A \in \mathbb{A}$  such that  $W_A^{\delta}(x) - W^{\delta}(x) \ge \delta$ , it is not difficult to see that  $\rho_A^{\varepsilon,\delta}(x) \le \exp\{-\delta/\varepsilon\}$ .

**Remark 7.4.** In general, the parameters  $\varepsilon$  and  $\delta$  can depend on n, and we will write them as  $\varepsilon_n$  and  $\delta_n$ .

#### 7.3 Importance sampling algorithm

For each  $A \in \mathbb{A}$ , the vector  $2\bar{\alpha}^A$  through Proposition 6.2 defines a new set of jump intensities of form

$$\bar{r}^*(x, 2\bar{\alpha}^A)[v] = r(x, v)e^{-\langle \bar{\alpha}^A, v \rangle}$$

The change of measure used in our state-dependent importance sampling scheme corresponds to a stochastic transition kernel of form

$$\bar{\Theta}^n[dt, v|x] = \bar{R}^n(x)e^{-\bar{r}^n(x,v)}dt,$$

where for  $x \neq 0$ 

$$\bar{r}^n(x,v) \doteq \sum_{A \in \mathbb{A}} \rho_A^{\varepsilon_n,\delta_n}(x)\bar{r}^*(x,2\bar{\alpha}^A)[v], \quad \bar{R}^n(x) \doteq \sum_{v \in \mathbb{V}} \bar{r}^n(x,v).$$

The values at x = 0 are unimportant, and for simplicity we use the original jump intensities, that is,

$$\bar{r}^n(0,v) \doteq r(0,v), \quad \bar{R}^n(0) \doteq \sum_{v \in \mathbb{V}} r(0,v).$$

The corresponding importance sampling estimator  $\hat{p}_n$  is just as defined in (5.4). Under this change of measure, the state process Q is again a continuous time Markov jump process with jump intensity  $\bar{r}^n(x,v)$  from state nx to nx + v whenever  $nx \in \mathbb{Z}_+^d$ . Note that the calculation of  $\bar{\Theta}^n$  is simple since  $\rho_A^{\varepsilon_n,\delta_n}$  and  $\bar{r}^*$  can be easily obtained.

**Remark 7.5.** The estimator  $\hat{p}_n$  described here is not exactly the one that will be used in the numerical experiments. Indeed, we will use a closely related, easier-to-implement, discrete time version of  $\hat{p}_n$ , say  $\bar{p}_n$ . The precise definition of  $\bar{p}_n$  can be found in Section 9. It turns out that  $\bar{p}_n$  is the expectation of  $\hat{p}_n$  conditioned on a suitable  $\sigma$ -algebra, and therefore  $\bar{p}_n$  is unbiased and has smaller second moment than  $\hat{p}_n$ . The motivation for focusing so far on  $\hat{p}_n$  is that it is more suitable for the asymptotic analysis.

# 8 Asymptotic Efficiency

In this section we show that the importance sampling estimator  $\hat{p}_n$  is asymptotically optimal under suitable conditions. We will need the following result, which essentially says that the return time to the origin is exponentially bounded. Recall that  $\{T_1, T_2, \ldots\}$  are the random jump times of the process Q and  $T_{N^0}$  by (5.3) is the first time the process returns to the origin.

**Lemma 8.1.** There exist positive constants c and k such that for every  $q \in \mathbb{Z}^d_+$ 

$$\log \mathbb{E}^{\mathbb{P}}[\exp\{cT_{N^{0}}\}|Q(0) = q] \le k(1 + ||q||).$$

The proof is deferred to the appendix.

**Proposition 8.2.** Suppose that  $\delta_n \to 0$ ,  $\varepsilon_n/\delta_n \to 0$ , and  $n\varepsilon_n \to \infty$ . Then for any sequence  $\{q_n\} \subset \mathbb{Z}^d_+$  such that  $q_n/n \to 0$ , we have the upper bound on the second moment of the importance sampling estimator

$$\limsup_{n} \frac{1}{n} \log \mathbb{E}^{\mathbb{P}}[\hat{p}_n | Q(0) = q_n] \le -2\gamma.$$

**Proof.** Let  $x_n = q_n/n$ . Then  $x_n \to 0$ . Throughout the proof, to ease notation, we denote  $W(x) \doteq W^{\varepsilon_n,\delta_n}(x)$ ,  $\rho_A(x) \doteq \rho_A^{\varepsilon_n,\delta_n}(x)$ , and  $E_{x_n}^{\mathbb{P}}[\cdot] \doteq E^{\mathbb{P}}[\cdot|Q(0) = nx_n]$ . Furthermore, note that all we need to show is that the upper bound holds for  $x_n \neq 0$  since we can write

$$E_0^{\mathbb{P}}[\hat{p}_n] = \sum_{i=1}^d \frac{\lambda_i}{\lambda_1 + \dots + \lambda_d} E^{\mathbb{P}}[\hat{p}_n | Q(0) = e_i].$$

We will make use of Lemma A.1, which identifies certain saddle points and is a generalization of Proposition 6.2. Assume from now on that  $x_n \neq 0$ . Fix an arbitrary s > 2. Since eventually we will send s to 2, we assume without loss of generality that  $s \in (2,3]$ . Consider  $L_s$  as defined in (A.1). We have, for every  $\hat{r} \in \mathcal{R}$ ,

$$L_s(x, DW(x); \bar{r}^n, \hat{r}) = \frac{s}{2} \sum_{v \in \mathbb{V}} \hat{r}(v) \langle DW(x), v \rangle + s \sum_{v \in \mathbb{V}} r(x, v) \ell\left(\frac{\hat{r}(v)}{r(x, v)}\right) - (s-1) \sum_{v \in \mathbb{V}} \bar{r}^n(x, v) \ell\left(\frac{\hat{r}(v)}{\bar{r}^n(x, v)}\right).$$

Note that by definition

$$DW(x) = \sum_{A \in \mathbb{A}} \rho_A(x) \cdot 2\bar{\alpha}^A, \quad \bar{r}^n(x,v) = \sum_{A \in \mathbb{A}} \rho_A(x) \cdot \bar{r}^*(x,2\bar{\alpha}^A)[v].$$

It follows by the concavity of the logarithmic function that

$$L_s(x, DW(x); \bar{r}^n, \hat{r}) \ge \sum_{A \in \mathbb{A}} \rho_A(x) L_s(x, 2\bar{\alpha}^A; \bar{r}^*(x, 2\bar{\alpha}^A), \hat{r}).$$

In particular,

$$\inf_{\hat{r}\in\mathcal{R}} L_s(x, DW(x); \bar{r}^n, \hat{r}) \ge \sum_{A\in\mathbb{A}} \rho_A(x) \inf_{\hat{r}\in\mathcal{R}} L_s(x, 2\bar{\alpha}^A; \bar{r}^*(x, 2\bar{\alpha}^A), \hat{r}).$$

By Lemma A.1,

$$\inf_{\hat{r}\in\mathcal{R}} L_s(x, 2\bar{\alpha}^A; \bar{r}^*(x, 2\bar{\alpha}^A), \hat{r}) = \frac{s}{2} \mathbb{H}_j(2\bar{\alpha}^A)$$

where  $j = \max\{i : i \in \pi(x)\}$ . Therefore, it follows from Lemma 7.3 that

$$\inf_{\hat{r}\in\mathcal{R}} L_s(x, DW(x); \bar{r}^n, \hat{r}) \ge -\frac{s}{2} K e^{-\delta_n/\varepsilon_n}$$

for some constant K that only depends on the system parameters  $\lambda_i, \mu_i$ , etc. However, by straightforward calculation (we omit the details)

$$\inf_{\hat{r}\in\mathcal{R}} L_s(x, DW(x); \bar{r}^n, \hat{r}) = (s-1)[R(x) - \bar{R}^n(x)] + \sum_{v\in\mathbb{V}} r(x, v) \left[ 1 - e^{-s\langle DW(x), v \rangle/2} \left( \frac{r(x, v)}{\bar{r}^n(x, v)} \right)^{s-1} \right]$$

.

Furthermore, by straightforward calculation it can be shown that every component of the Hessian matrix  $D^2W(x)$  is uniformly bounded by  $C/\varepsilon_n$  for some constant C that only depends on the system parameters. It follows from Taylor's expansion that

$$|\langle DW(x), v \rangle - n \left[ W(x + v/n) - W(x) \right]| \le \frac{\bar{C}}{n\varepsilon_n}$$

for some constant  $\overline{C}$ , which again depends only on the system parameters. Observing that  $r(x, v)/\overline{r}^n(x, v)$  is uniformly bounded, it follows easily that, for all  $s \in (2, 3]$  and  $x \in \mathbb{R}^d_+$ ,

$$\begin{aligned} h_n(x) &\doteq (s-1)[R(x) - \bar{R}^n(x)] \\ &+ \sum_{v \in \mathbb{V}} r(x,v) \left[ 1 - e^{-sn(W(x+v/n) - W(x))/2} \left( \frac{r(x,v)}{\bar{r}^n(x,v)} \right)^{s-1} \right] \\ &\geq -\beta_n, \end{aligned}$$

where

$$\beta_n \doteq 2Ke^{-\delta_n/\varepsilon_n} + \bar{K} \left[ e^{\bar{C}/(2n\varepsilon_n)} - 1 \right]$$

for some constants K and  $\overline{K}$  that only depend on the system parameters. Let  $J^n(t) = \min\{j : T_j \ge t\}$ , and let

$$\hat{p}_n(t) \doteq \exp\left\{\int_0^t [\bar{R}(Q(s)/n) - R(Q(s)/n)]ds + \sum_{j=1}^{J^n(t)} \log \frac{r(Q(T_{j-1})/n, v_j)}{\bar{r}(Q(T_{j-1})/n, v_j)}\right\},\$$

so that  $\hat{p}_n = \hat{p}_n(T_{N^n})$ . Note that two types of jump processes appear in the exponent, namely Q and the sum of log terms up to  $J^n(t)$ . Although the sum is slightly non-standard in that the size of the jump depends on several factors (including the state of Q at the time of the last jump), one can construct a generalized Itô formula for  $\hat{p}_n(t)$ .

Now consider the nonnegative process

$$M^{n}(t) \doteq e^{-\beta_{n}t - snW(Q(t)/n)/2} \hat{p}_{n}^{s-1}(t).$$
(8.1)

By the definitions of  $M^n$  and  $\hat{p}_n$  and the generalized Itô formula, the process

$$M^{n}(t) + \int_{0}^{t} M^{n}(u) \left[h_{n}(Q(u)/n) + \beta_{n}\right] du$$

is a local martingale. Since  $h_n(x) \ge -\beta_n$ , it follows that  $M^n(t)$  is local supermartingale, whence a true supermartingale due to its non-negativity.

Consequently, the optional sampling theorem implies that, with  $N \doteq N^n \wedge N^0$ ,

$$E_{x_n}^{\mathbb{P}}[M^n(T_N)] \le M^n(0) = e^{-snW(x_n)/2}.$$

Observing that  $Q(T_N)/n \notin D$  on the set  $\{N^n < N^0\}$ , while  $\hat{p}_n = 0$  on the set  $\{N^n \ge N^0\}$ , and that the boundary condition  $W(x) \le 0$  holds for all  $x \notin D$ , it follows that w.p.1

$$M^n(T_N) \ge e^{-\beta_n T_N} \hat{p}_n^{s-1}.$$

Therefore,

$$E_{x_n}^{\mathbb{P}}\left[e^{-\beta_n T_N}\hat{p}_n^{s-1}\right] \le e^{-snW(x_n)/2}.$$

Using Holder's inequality

$$E_{x_n}^{\mathbb{P}}[\hat{p}_n] \leq \left( E_{x_n}^{\mathbb{P}} \left[ e^{-\beta_n T_N} \hat{p}_n^{s-1} \right] \right)^{\frac{1}{s-1}} \left( E_{x_n}^{\mathbb{P}} \left[ e^{\frac{\beta_n}{s-2} T_N} \mathbb{1}_{\{N^n < N^0\}} \right] \right)^{\frac{s-2}{s-1}} \\
 \leq e^{-\frac{s}{2(s-1)} n W(x_n)} \left( E_{x_n}^{\mathbb{P}} \left[ e^{\frac{\beta_n}{s-2} T_{N^0}} \right] \right)^{\frac{s-2}{s-1}}.$$

Note that the above inequality is true for all  $s \in (2,3]$ . In particular, let c be the constant given by Lemma 8.1, and let  $s = s_n$  where

$$\frac{\beta_n}{s_n-2} = c$$
 or  $s_n = 2 + \frac{\beta_n}{c}$ 

Note that by assumption  $\beta_n \to 0$ , thus  $s_n \in (2,3]$  for *n* large enough. Hence by Lemma 8.1

$$E_{x_n}^{\mathbb{P}}[\hat{p}_n] \le e^{-\frac{s_n}{2(s_n-1)}nW(x_n)} \left( E_{x_n}^{\mathbb{P}} \left[ e^{cT_{N^0}} \right] \right)^{\frac{s_n-2}{s_n-1}} \le e^{-\frac{s_n}{2(s_n-1)}nW(x_n)} e^{\frac{k(s_n-2)}{s_n-1}(1+\|x_n\|)}$$

Therefore,

$$\frac{1}{n}\log E_{x_n}^{\mathbb{P}}[\hat{p}_n] \le -\frac{s_n}{2(s_n-1)}W(x_n) + \frac{1}{n}\frac{k(s_n-2)}{s_n-1}(1+\|x_n\|)$$

Note that  $s_n \to 2$  and  $||x_n|| \to 0$  as  $n \to \infty$ , and that

$$W^n(0) \ge 2\gamma - \log |\mathbb{A}| \cdot \varepsilon_n - C_d \delta_n$$

Now letting  $n \to 0$  we have

$$\limsup_{n} \frac{1}{n} \log E_{x_n}^{\mathbb{P}}[\hat{p}_n] \le -2\gamma.$$

This completes the proof.

**Theorem 8.3.** The exponential decay rate of  $\{p_n\}$  is  $\gamma$ , that is,

$$\lim_{n} \frac{1}{n} \log p_n = -\gamma$$

**Proof.** It follows (5.1), the unbiasedness of  $\hat{p}_n$ , and Jensen's inequality that  $E_0^{\mathbb{P}}[\hat{p}_n] \geq p_n^2$ . Therefore, by Proposition 8.2 the upper bound

$$\limsup_{n} \frac{1}{n} \log p_n \le -\gamma$$

holds. It remains to show the lower bound. Thanks to Theorem 4.2, it suffices to show that

$$\liminf_{n} \frac{1}{n} \log p_n \ge -\int_0^\tau L(\phi(t), \dot{\phi}(t)) dt$$
(8.2)

for any absolutely continuous  $\phi : \mathbb{R}_+ \to \mathbb{R}^d_+$  such that  $\phi(0) = 0$  and  $\tau \doteq \inf\{t \ge 0 : \phi(t) \in \partial\} < \infty$ . Note that due to the non-negativity of L it suffices to consider  $\phi$  such that  $\phi(t) \ne 0$  for all  $t \in (0, \tau]$ . For any  $T \ge 0$  denote by  $\mathcal{D}[0, T]$  the collection of cadlag functions on [0, T] taking values in  $\mathbb{R}^d_+$ , equipped with the Skorohod topology. Let  $\mathbf{1} \doteq (1, 1, \ldots, 1)$  and

$$\varphi(t) \doteq \begin{cases} \phi(t) & \text{if } t \leq \tau, \\ \phi(\tau) + (t - \tau) \cdot \mathbf{1} & \text{if } t > \tau. \end{cases}$$

For any  $\varepsilon, \delta > 0$ , define

$$B^{\varepsilon}(\varphi; \delta) \doteq \{ \psi \in \mathcal{D}[0, \tau] : \| \psi(t) - \varphi(t + \varepsilon) \| < \delta \text{ for all } 0 \le t \le \tau \}.$$

Observing that  $\dot{\varphi}(t) = \mathbf{1}$  for  $t > \tau$  and that  $L^{A}(\mathbf{1})$  is finite for any  $A \subset \{1, \ldots, d\}$ , it follows that

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\tau+\varepsilon} L(\varphi(t), \dot{\varphi}(t)) \, dt = \int_{0}^{\tau} L(\phi(t), \dot{\phi}(t)) \, dt.$$
(8.3)

For every  $\varepsilon$ , there exists a  $\delta = \delta(\varepsilon)$  such that for every  $\psi \in B^{\varepsilon}(\varphi; \delta), \psi(t) \neq 0$ for all  $t \in [0, \tau]$  and  $\psi(\tau) \notin D$ . Now for each *n* define

$$x_n \doteq \frac{1}{n} \lfloor n\phi(\varepsilon) \rfloor,$$

where the integer part is applied component-wise. Then

 $p_n \ge \mathbb{P}_0(X^n \text{ hits } x_n \text{ before } 0)\mathbb{P}_{x_n}(X^n \text{ exits } D \text{ before hitting } 0).$ 

However, note that  $B^{\varepsilon}(\varphi; \delta)$  is open under the Skorohod topology [12, Appendix A] since  $\varphi$  is continuous, and that  $\psi(t) \doteq \varphi(t + \varepsilon)$  for  $t \in [0, \tau]$  is an element of  $B^{\varepsilon}(\varphi; \delta)$ . Moreover  $x_n \to \phi(\varepsilon)$ , and  $\{X^n\}$  satisfies the sample path large deviation principle with local rate function L [5]. It follows that

$$\liminf_{n} \frac{1}{n} \log \mathbb{P}_{x_{n}}(X^{n} \text{ exits } D \text{ before hitting } 0)$$

$$\geq \liminf_{n} \frac{1}{n} \log \mathbb{P}_{x_{n}}(X^{n} \in B^{\varepsilon}(\varphi; \delta))$$

$$\geq -\inf \left\{ \int_{0}^{\tau} L(\psi(t), \dot{\psi}(t)) dt : \psi \in B^{\varepsilon}(\varphi; \delta) \right\}$$

$$\geq -\int_{0}^{\tau} L(\varphi(t+\varepsilon), \dot{\varphi}(t+\varepsilon)) dt.$$
(8.4)

Furthermore, it is easy to see that

$$\mathbb{P}_0(\text{hitting } x_n \text{ before } 0) \ge \prod_{i=1}^d \left(\frac{\lambda_i}{\sum_{j=1}^d (\lambda_i + \mu_i)}\right)^{(nx_n)_i}$$

Therefore

$$\liminf_{n} \frac{1}{n} \log \mathbb{P}_{0}(\text{hitting } x_{n} \text{ before } 0)$$

$$\geq \sum_{i=1}^{d} \lim \frac{1}{n} (nx_{n})_{i} \log \frac{\lambda_{i}}{\sum_{j=1}^{d} (\lambda_{i} + \mu_{i})}$$

$$= \sum_{i=1}^{d} (\phi(\varepsilon))_{i} \log \frac{\lambda_{i}}{\sum_{j=1}^{d} (\lambda_{i} + \mu_{i})}.$$
(8.5)

Combining (8.3), (8.4), and (8.5), and letting  $\varepsilon \to 0$ , we arrive at the desired inequality (8.2).

The following result is immediate from Proposition 8.2 and Theorem 8.3.

**Corollary 8.4.** Suppose that  $\delta_n \to 0$ ,  $\varepsilon_n / \delta_n \to 0$ , and  $n\varepsilon_n \to \infty$ . Then the importance sampling estimator  $\hat{p}_n$  is asymptotically optimal. That is,

$$\lim_{n} \frac{1}{n} \log \mathbb{E}_0^{\mathbb{P}}[\hat{p}_n] = -2\gamma.$$

# 9 Numerical Experiments

For ease of computation the numerical results are obtained using the embedded discrete time Markov chain  $Z = \{Z(j) = Q(T_j), j \ge 0\}$ . This means that given that the current state Z(j) is nx, the next jump  $\Delta_{j+1}$  is chosen according to

$$\mathbb{Q}(\Delta_{j+1} = v) = \bar{r}^n(x, v) / \bar{R}^n(x).$$

The actual importance sampling estimator is then

$$\bar{p}_n \doteq \mathbb{1}_{\{Z(N) \notin D\}} \prod_{j=0}^{N-1} \frac{r(Z(j)/n, \Delta_{j+1})/R(Z(j)/n)}{\bar{r}^n(Z(j)/n, \Delta_{j+1})/\bar{R}^n(Z(j)/n)},$$
(9.1)

where

$$N = \inf\{k \ge 1 : Q(T_k) \in \partial \text{ or } Q(T_k) = 0\}.$$

By referring to the definition (5.4), a simple calculation shows that

$$E^{\mathbb{Q}}[\hat{p}_n|Q(T_1),\ldots,Q(T_N)]=\bar{p}_n.$$

Therefore the asymptotic optimality of  $\bar{p}_n$  follows from the asymptotic optimality of  $\hat{p}_n$ .

In the tables below we present results from three different systems in 2, 4, and 6 dimensions. For each system we consider n = 20, 50, and 80. All results are achieved using 20,000 samples.

	n = 20	n = 50	n = 80
Theoretical value	$1.90 \times 10^{-5}$	$4.36 \times 10^{-13}$	$8.31 \times 10^{-21}$
Estimate	$1.94 \times 10^{-5}$	$4.36 \times 10^{-13}$	$8.18 \times 10^{-21}$
Std. Err.	$0.04 \times 10^{-5}$	$0.12 \times 10^{-13}$	$0.32 \times 10^{-21}$
95% C.I.	$[1.86, 2.02] \times 10^{-5}$	$[4.12, 4.59] \times 10^{-13}$	$[7.55, 8.81] \times 10^{-32}$

Table 1.  $(\lambda_1, \lambda_2) = (1, 2), (\mu_1, \mu_2) = (3, 4), \text{ and } (c_1, c_2) = (1/2, 1)$ 

	n = 20	n = 50	n = 80
Theoretical value	$5.62 \times 10^{-9}$	$1.54 \times 10^{-14}$	$7.01 \times 10^{-23}$
Estimate	$5.52 \times 10^{-9}$	$1.51 \times 10^{-14}$	$6.91 \times 10^{-23}$
Std. Err.	$0.27 \times 10^{-9}$	$0.09 \times 10^{-14}$	$0.33 \times 10^{-23}$
95% C.I.	$[4.99, 6.04] \times 10^{-9}$	$[1.33, 1.69] \times 10^{-14}$	$[6.26, 7.56] \times 10^{-23}$

Table 2.  $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1, 2, 2, 4), (\mu_1, \mu_2, \mu_3, \mu_4) = (5, 12, 10, 15), \text{ and } (c_1, c_2, c_3, c_4) = (1/2, 1, 1, 1)$ 

	n = 20	n = 50	n = 80
Theoretical value	$2.1 \times 10^{-8}$	$4.2 \times 10^{-13}$	$3.7 \times 10^{-20}$
Estimate	$2.21 \times 10^{-8}$	$4.21 \times 10^{-13}$	$3.75 \times 10^{-20}$
Std. Err.	$0.08 \times 10^{-8}$	$0.15 \times 10^{-13}$	$0.14 \times 10^{-20}$
95% C.I.	$[2.05, 2.37] \times 10^{-8}$	$[3.91, 4.50] \times 10^{-13}$	$[3.48, 4.02] \times 10^{-20}$

Table 3.  $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = (1, 2, 2, 3, 1, 8), (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6) = (10, 15, 16, 16, 15, 24), \text{ and } (c_1, c_2, c_3, c_4, c_5, c_6) = (1/2, 1, 1, 1, 1, 1/3)$ 

For all cases the values of the constants  $C_{|A|}$  are determined using the formulas in (7.2) and (7.3). The constants  $\delta_n$  and  $\varepsilon_n$  are determined by the formulas  $\varepsilon_n = \frac{1}{5 \log n}$  and  $\delta_n = \varepsilon_n \log \varepsilon_n$ . The results are robust with respect to the choice of the parameters  $\varepsilon_n$  and  $\delta_n$ . For example  $\varepsilon_n = c/\sqrt{n}$  yields similar quality results for varying values of c.

For the two and four dimensional systems the exact values are obtained by solving the linear system that arises from a first step analysis. This approach is computationally infeasible for the six dimensional system, thus for that system the exact value is obtained by running the importance sampling algorithm for 2 million iterations.

### A Appendix. Collection of Proofs

#### A.1 Proof of Lemma 6.1

We will consider two cases separately.

Case 1.  $c + s[\theta] \leq 0$ . It is not difficult to see that the integral on the left-hand-side (LHS) is  $\infty$ . Therefore, it suffices to show that the infimum on the right-hand-side (RHS) is  $-\infty$ . Fix an arbitrary  $\varepsilon > 0$  and consider  $\hat{\theta}_{\varepsilon} \in \mathbb{R}$  such that

$$\hat{\theta}_{\varepsilon}(v) \doteq \varepsilon e^{-h(v)} \theta(v).$$

Then simple algebra yields

RHS = 
$$\frac{c + s[\theta]}{\varepsilon \sum_{v \in \mathbb{V}} e^{-h(v)} \theta(v)} + \log \varepsilon - 1.$$

Letting  $\varepsilon \to 0$ , since  $c + s[\theta] \le 0$ , we conclude that the infimum on the RHS does equal  $-\infty$ .

Case 2.  $c + s[\theta] > 0$ . Abusing notation, for each  $\theta \in \mathcal{R}$ , we define the probability measure induced by  $\theta$  on  $\mathbb{R}_+ \times \mathbb{V}$  by

$$\mathbb{P}_{\theta}(dt, v) = e^{-s[\theta]t}\theta(v)dt.$$

Then we can write

LHS = 
$$-\log \int_{\mathbb{R}_+} \sum_{v \in \mathbb{V}} e^{-[ct+h(v)]} \mathbb{P}_{\theta}(dt, v).$$

Further abusing the notation, let  $\mathcal{P}$  denote the collection of all probability measures on  $\mathbb{R}_+ \times \mathbb{V}$ . Then by a slight extension of [4, Proposition 1.4.2], namely [7, Lemma A.1], we have

LHS = 
$$\inf_{\mathbb{Q}\in\mathcal{P}}\left[\int_{\mathbb{R}^+} \sum_{v\in\mathbb{V}} [ct+h(v)]\mathbb{Q}(dt,v) + R(\mathbb{Q}||\mathbb{P}_{\theta})\right]$$

with the minimizing  $\mathbb{Q}^*$  given by

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}_{\theta}} = ke^{-ct-h(v)},$$

where k is the normalizing constant. It is not difficult to see that the minimizing  $\mathbb{Q}^*$  is an element of  $\{\mathbb{P}_{\hat{\theta}} : \hat{\theta} \in \mathcal{R}\}$ . Therefore,

$$LHS = \inf_{\hat{\theta} \in \mathcal{R}} \left[ \int_{\mathbb{R}_+} \sum_{v \in \mathbb{V}} [ct + h(v)] \mathbb{P}_{\hat{\theta}}(dt, v) + R(\mathbb{P}_{\hat{\theta}} \| \mathbb{P}_{\theta}) \right]$$

But it is straightforward to verify that

$$\int_{\mathbb{R}_+} \sum_{v \in \mathbb{V}} [ct + h(v)] \mathbb{P}_{\hat{\theta}}(dt, v) = \frac{1}{s[\hat{\theta}]} \left[ c + \sum_{v \in \mathbb{V}} h(v) \hat{\theta}(v) \right]$$

and

$$R(\mathbb{P}_{\hat{\theta}} \| \mathbb{P}_{\theta}) = \frac{1}{s[\hat{\theta}]} \sum_{v \in \mathbb{V}} \theta(v) \ell\left(\frac{\hat{\theta}(v)}{\theta(v)}\right).$$

#### A.2 Proof of Proposition 6.2

We will prove a stronger version of Proposition 6.2, which will be useful in the asymptotic analysis of the importance sampling estimators. Fix an arbitrary  $s \ge 1$ . We define, for any  $x \in \mathbb{R}^d_+$  and  $\bar{r}, \hat{r} \in \mathcal{R}$ ,

$$L_{s}(x,\alpha;\bar{r},\hat{r}) \doteq \frac{s}{2} \sum_{v \in \mathbb{V}} \hat{r}(v) \langle \alpha, v \rangle + s \sum_{v \in \mathbb{V}} r(x,v) \ell\left(\frac{\hat{r}(v)}{r(x,v)}\right) - (s-1) \sum_{v \in \mathbb{V}} \bar{r}(v) \ell\left(\frac{\hat{r}(v)}{\bar{r}(v)}\right)$$
(A.1)

and

$$\mathbb{H}_s(x,\alpha) = \sup_{\bar{r} \in \mathcal{R}} \inf_{\hat{r} \in \mathcal{R}} L_s(x,\alpha;\bar{r},\hat{r}).$$

Note that  $\mathbb{H}$  is the special case of  $\mathbb{H}_s$  with s = 2. The following result subsumes Proposition 6.2.

**Lemma A.1.** Assume that  $s \ge 1$ . For any  $x \in \mathbb{R}^d_+ \setminus \{0\}$  and  $\alpha \in \mathbb{R}^d$ ,

- 1.  $\mathbb{H}_s(x,\alpha) = -sH^{(j)}(-\alpha/2) = s\mathbb{H}_j(\alpha)/2$  where  $j = \max\{i : i \in \pi(x)\}$ . In particular, for every x,  $\mathbb{H}_s(x, \cdot)$  is concave.
- 2. The saddle point of  $L_s$  is independent of s and takes the form

$$\bar{r}^*(x,\alpha)[v] = \hat{r}^*(x,\alpha)[v] = r(x,v)e^{-\langle \alpha,v\rangle/2}.$$

*Proof.* We first argue that  $(\bar{r}^*(x,\alpha), \hat{r}^*(x,\alpha))$  is a saddle point, that is,

$$L_s(x,\alpha;\bar{r},\hat{r}^*(x,\alpha)) \le L_s(x,\alpha;\bar{r}^*(x,\alpha),\hat{r}^*(x,\alpha)) \le L_s(x,\alpha;\bar{r}^*(x,\alpha),\hat{r})$$

for all  $\bar{r}, \hat{r} \in \mathcal{R}$ . The first inequality is trivial due to that  $s \geq 1$ , the nonnegativity of  $\bar{r}$  and function  $\ell$ , and that  $\ell(z) = 0$  if and only if z = 1. As for the second inequality, straightforward calculation yields that

$$L_s(x,\alpha;\bar{r}^*(x,\alpha),\hat{r}) = \frac{1}{2} \sum_{v \in \mathbb{V}} \hat{r}(v) \langle \alpha, v \rangle + \sum_{v \in \mathbb{V}} r(x,v) \ell\left(\frac{\hat{r}(v)}{r(x,v)}\right) - (s-1) H^{(j)}(-\alpha/2).$$

By elementary calculus, the right-hand-side is minimized at  $\hat{r} = \hat{r}^*(x, v)$ . Therefore, the second inequality holds, and  $(\bar{r}^*(x, \alpha), \hat{r}^*(x, \alpha))$  is a saddle point. In particular,

$$\mathbb{H}_s(x,\alpha) = L_s(x,\alpha;\bar{r}^*(x,\alpha),\hat{r}^*(x,\alpha)) = -sH^{(j)}(-\alpha/2) = s\mathbb{H}_j(\alpha)/2.$$

This completes the proof.

#### A.3 Proof of Lemma 7.2.

Fix an arbitrary  $x \in \mathbb{R}^d_+ \setminus \{0\}$ . Let  $w^* \doteq \max\{c_i x_i : i = 1, \dots, d\}$  and

$$x^* \doteq w^* \bar{c} = (w^*/c_1, w^*/c_2, \dots, w^*/c_d).$$

Then  $x_i - x_i^* \leq 0$  for all *i* with equality if and only if  $i \in \pi(x)$ . Moreover, for any nonempty subset  $F \subset \{1, \ldots, d\}$ , the definition (7.1) and Lemma 7.1 imply that

$$\begin{aligned} \langle 2\bar{\alpha}^F, x \rangle &= \langle 2\bar{\alpha}^F, x^* \rangle + \langle 2\bar{\alpha}^F, x - x^* \rangle \\ &= 2w^* \langle 2\bar{\alpha}^F, \bar{c} \rangle + \sum_{i \notin \pi(x)} 2\bar{\alpha}_i^F(x_i - x_i^*) \\ &= -2w^*\gamma + \sum_{i \in F \setminus \pi(x)} 2\bar{\alpha}_i^F(x_i - x_i^*). \end{aligned}$$

It follows that for any  $A \subset \{1, \ldots, d\}$ 

$$W_{A}^{\delta}(x) - W_{F}^{\delta}(x) = \sum_{i \in A \setminus \pi(x)} 2\bar{\alpha}_{i}^{A}(x_{i} - x_{i}^{*}) - \sum_{i \in F \setminus \pi(x)} 2\bar{\alpha}_{i}^{F}(x_{i} - x_{i}^{*}) + (C_{|F|} - C_{|A|})\delta.$$
(A.2)

Since  $W_F^{\delta}(x) \ge W^{\delta}(x)$  for any F, we have by assumption

$$W_A^{\delta}(x) - W_F^{\delta}(x) \le W_A^{\delta}(x) - W^{\delta}(x) < \delta.$$

In particular, the above inequality holds for  $F = \pi(x)$  and  $F = A \cup \pi(x)$ and (A.2) then becomes

$$\delta > \sum_{i \in A \setminus \pi(x)} 2\bar{\alpha}_i^A(x_i - x_i^*) + (C_{|\pi(x)|} - C_{|A|})\delta, \quad (A.3)$$
  
$$\delta > \sum_{i \in A \setminus \pi(x)} 2(\bar{\alpha}_i^A - \bar{\alpha}_i^{A \cup \pi(x)})(x_i - x_i^*) + (C_{|A \cup \pi(x)|} - C_{|A|})\delta. \quad (A.4)$$

We now argue by contradiction that  $|A \cup \pi(x)| = |A|$  or  $\pi(x) \subset A$ . To this end, assume that |A| = m and  $|A \cup \pi(x)| = m + j$  with j > 0. It follows easily from the definition of  $\{a_k^*\}$  that for every  $i \in A$ 

$$\frac{\bar{\alpha}_i^{A\cup\pi(x)}}{\bar{\alpha}_i^A} \le a_m^* a_{m+1}^* \cdots a_{m+j-1}^*.$$

Note that  $a_k^* \geq 1$  for all k, and  $\bar{\alpha}_i^A < 0$  and  $x_i - x_i^* < 0$  for all  $i \in A \setminus \pi(x)$ . Therefore, it follows from (A.4) and (A.3), that

$$\begin{aligned} (C_{|A\cup\pi(x)|} - C_{|A|} - 1)\delta &< \sum_{i\in A\setminus\pi(x)} 2(\bar{\alpha}_i^{A\cup\pi(x)} - \bar{\alpha}_i^A)(x_i - x_i^*) \\ &\leq (a_m^* a_{m+1}^* \cdots a_{m+j-1}^* - 1) \sum_{i\in A\setminus\pi(x)} 2\bar{\alpha}_i^A(x_i - x_i^*) \\ &< (a_m^* a_{m+1}^* \cdots a_{m+j-1}^* - 1)(C_{|A|} - C_{|\pi(x)|} + 1)\delta \\ &\leq (a_m^* a_{m+1}^* \cdots a_{m+j-1}^* - 1)(C_{|A|} + 1)\delta, \end{aligned}$$

or

$$C_{m+j} - C_m - 1 < (a_m^* a_{m+1}^* \cdots a_{m+j-1}^* - 1)(C_m + 1),$$

or

$$C_{m+j} < a_m^* a_{m+1}^* \cdots a_{m+j-1}^* (C_m + 1).$$

But this is impossible since

$$C_{m+j} = a_{m+j-1}^* (1 + C_{m+j-1})$$
  
>  $a_{m+j-1}^* C_{m+j-1}$   
=  $a_{m+j-1}^* a_{m+j-2}^* (1 + C_{m-j+2})$   
>  $\cdots$   
=  $a_m^* a_{m+1}^* \cdots a_{m+j-1}^* (C_m + 1),$ 

a contradiction. Thus j = 0 and  $|A \cup \pi(x)| = |A|$ , and whence  $\pi(x) \subset A$ .

For every  $x \notin D$ , observe that  $x_i \geq 1/c_i$  for all  $i \in \pi(x)$ . Therefore, by Lemma 7.1 and the definition of  $W^{\delta}$ ,

$$W^{\delta}(x) \le W^{\delta}_{\pi(x)}(x) \le \langle 2\bar{\alpha}^{\pi(x)}, x \rangle + 2\gamma \le \langle 2\bar{\alpha}^{\pi(x)}, \bar{c} \rangle + 2\gamma = 0.$$

We complete the proof.

#### A.4 Proof of Lemma 8.1

Assume for now that  $Q(0) = q \neq 0$ . To ease notation, unless specified, we simply denote

$$\mathbb{E}^{\mathbb{P}}[\cdot] \doteq \mathbb{E}^{\mathbb{P}}[\cdot|Q(0) = q].$$

We first show that  $T_{N^0}$  is finite with probability one. Define a vector

$$d = (1/\mu_1, 1/\mu_2, \dots, 1/\mu_d)$$

and let

$$\varepsilon \doteq 1 - \sum_{i=1}^d \frac{\lambda_i}{\mu_i} > 0.$$

Consider the process  $Y = \{Y(t)\}$  where

$$Y(t) \doteq \langle Q(t), d \rangle + \varepsilon t.$$

It is easy to check that the compensation process for  $\{\langle Q(t), d \rangle\}$  is

$$\int_0^t \left[ \sum_{j=1}^d \lambda_j \langle d, e_j \rangle - \sum_{j=1}^d \mu_j \langle d, -e_j \rangle \mathbb{1}_{\{j=\max \pi(Q(t))\}} \right] \, ds = -\varepsilon t.$$

Therefore, Y is a local martingale. Since Y is a non-negative, it is a supermartingale. In particular, by the optional sampling theorem

$$\varepsilon \mathbb{E}^{\mathbb{P}}[T_{N^0}] \le \mathbb{E}^{\mathbb{P}}[Y(T_{N^0})] \le Y(0) = \langle q, d \rangle.$$

This implies that  $T_{N^0}$  has finite expectation. In particular, it is finite with probability one.

The proof for the exponential bound is based on a similar argument and the existence of a *strict* smooth subsolution. Fix an arbitrary  $\nu \in (0, 1)$  and for ease of notation let  $\alpha^* = \bar{\alpha}^{\{1, \dots, d\}}$ . We claim that there exists a c > 0such that the process  $M = \{M(t) : t \ge 0\}$  with

$$M(t) = \exp\{ct - \langle Q(t), \nu \alpha^* \rangle\}$$
(A.5)

is a supermartingale. Indeed, by the generalized Itô formula the process

$$M(t) - \int_0^t M(s)h(Q(s)) \, ds$$

is a local martingale with

$$h(z) \doteq c + \sum_{v \in \mathbb{V}} r(z, v) \left( \exp\{-\langle \nu \alpha^*, v \rangle\} - 1 \right) = c + H^{(i)}(-\nu \alpha^*)$$

with  $i = \max\{\pi(z)\}$ . It follows from Lemma 7.1 and the strict convexity of  $H^{(i)}$  that

$$H^{(i)}(-\nu\alpha^*) < 0$$

for all *i*. Therefore there exists a c > 0 such that  $h(z) \leq 0$ , which in turn implies that M is a local supermartingale. But M is nonnegative, whence a true supermartingale. By the optional sampling theorem,

$$\exp\{-\langle q, \nu \alpha^* \rangle\} \ge \mathbb{E}^{\mathbb{P}}[M(T_{N^0})] = \mathbb{E}^{\mathbb{P}}[\exp\{cT_{N^0}\}]$$

In particular, this implies that there exists some  $k_1 > 0$  such that for any  $q \neq 0$ ,

$$\log \mathbb{E}^{\mathbb{P}}[\exp\{cT_{N^0}\}|Q(0) = q] \le k_1 \|q\|.$$

It remains to show for the case where Q(0) = 0. By conditioning on the first jump of Q, it follows that

$$E^{\mathbb{P}}[\exp\{cT_{N^{0}}\}|Q(0)=0] = \sum_{i=1}^{d} \frac{\lambda_{i}}{\Lambda(\Lambda+c)} \mathbb{E}^{\mathbb{P}}[\exp\{cT_{N^{0}}\}|Q(0)=e_{i}] < \infty,$$

where  $\Lambda \doteq \lambda_1 + \dots + \lambda_d$ . Letting

$$k \doteq \max\{k_1, \log E^{\mathbb{P}}[\exp\{cT_{N^0}\} | Q(0) = 0]\},\$$

we complete the proof.

#### A.5 Proof of Theorem 4.2

The proof of Theorem 4.2 is essentially a verification argument, utilizing the smooth subsolution  $W^{\varepsilon,\delta}$ .

We first argue that, for any absolutely continuous function  $\phi : \mathbb{R}_+ \to \mathbb{R}^d_+$ with  $\phi(0) = 0$  and  $\tau = \inf\{t \ge 0 : \phi(t) \in \partial\} < \infty$ , the inequality

$$\gamma \le \int_0^\tau L(\phi(t), \dot{\phi}(t)) \, dt \tag{A.6}$$

holds. To this end, fix arbitrarily  $\varepsilon, \delta > 0$ , and to ease notation denote  $W = W^{\varepsilon,\delta}$ . Then by the definition of L,

$$L(\phi(t), \dot{\phi}(t)) = L^{\pi(\phi(t))}(\dot{\phi}(t)) = \sup_{\alpha \in \mathbb{R}^d} \left[ \langle \alpha, \dot{\phi}(t) \rangle - \max_{i \in \pi(\phi(t))} H^{(i)}(\alpha) \right].$$

In particular, thanks to Lemma 7.3 and that  $\mathbb{H}_i(\alpha) = -2H^{(i)}(-\alpha/2),$ 

$$\begin{split} L(\phi(t), \dot{\phi}(t)) &\geq \langle -DW(\phi(t))/2, \dot{\phi}(t) \rangle + \min_{i \in \pi(\phi(t))} \mathbb{H}_i(DW(\phi(t)))/2 \\ &\geq \langle -DW(\phi(t))/2, \dot{\phi}(t) \rangle - Ke^{-\delta/\varepsilon}/2. \end{split}$$

Integrating both sides from t = 0 to  $t = \tau$ , it follows that

$$\int_0^\tau L(\phi(t), \dot{\phi}(t)) \, dt \ge \frac{1}{2} \left[ W(\phi(0)) - W(\phi(\tau)) - K e^{-\delta/\varepsilon} \tau \right].$$

Observe that the boundary condition  $W(x) \leq 0$  holds for all  $x \in \partial$ , and that by inequality (7.6)

$$W(\phi(0)) = W(0) \ge W^{\delta}(0) - \varepsilon \log |\mathbb{A}| = 2\gamma - \varepsilon \log |\mathbb{A}| - C_d \delta.$$

Therefore,

$$\int_0^\tau L(\phi(t), \dot{\phi}(t)) \, dt \ge \gamma - \frac{1}{2} \left( \varepsilon \log |\mathbb{A}| + C_d \delta + K e^{-\delta/\varepsilon} \tau \right)$$

Letting  $\varepsilon, \delta \to 0$  but  $\delta/\varepsilon \to \infty$ , we arrive at the desired inequality (A.6).

In order to show the other direction, it suffices to construct an absolutely continuous function  $\phi^* : \mathbb{R}_+ \to \mathbb{R}^d_+$  such that  $\phi^*(0) = 0$  and  $\tau^* \doteq \inf\{t \ge 0 : \phi^*(t) \in \partial\} < \infty$  with

$$\int_{0}^{\tau^{*}} L(\phi^{*}(t), \dot{\phi}^{*}(t)) dt = \gamma.$$
 (A.7)

This  $\phi^*$  is actually an optimal trajectory for the calculus of variation problem. To this end, fix arbitrarily

$$A^* \in \operatorname{argmin}\left\{ \langle -\alpha^A, \bar{c} \rangle : A \in \mathbb{A} \right\}.$$

Consider the set valued function  $F^*:\mathbb{R}^d_+\to\mathbb{R}^d$  defined by

$$F^*(x) \doteq$$
 the convex hull of  $\{DH^{(i)}(-\alpha^{A^*}) : i \in \pi(x)\}$ 

for every  $x \in \mathbb{R}^d_+$ . Clearly  $F^*$  is upper semicontinuous since  $\pi$  is so. It follows the classical theory of differential inclusions that there exists an absolutely continuous solution to

$$\dot{\phi}(t) \in F^*(\phi(t)), \quad \phi(0) = 0.$$
 (A.8)

With  $\phi^*$  denoting this solution, we will argue that  $\phi^*$  satisfies all the desired properties. Note that by definition, we can write for almost every t

$$\dot{\phi}^{*}(t) = \sum_{i \in \pi(\phi^{*}(t))} \rho_{i}^{*}(t) DH^{(i)}(-\alpha^{A^{*}})$$
(A.9)

where  $\rho_i^*(t) \ge 0$  and

$$\sum_{i \in \pi(\phi^*(t))} \rho_i^*(t) = 1.$$
 (A.10)

Since

$$DH^{(i)}(-\alpha^{A^*}) = -\mu_i e^{\alpha_i^{A^*}} e_i + \sum_{j=1}^d \lambda_j e^{-\alpha_j^{A^*}} e_j,$$

it follows that

$$\dot{\phi}^*(t) = \sum_{j=1}^d \lambda_j e^{-\alpha_j^{A^*}} e_j - \sum_{i \in \pi(\phi^*(t))} \rho_i^*(t) \mu_i e^{\alpha_i^{A^*}} e_i.$$
(A.11)

Define for each  $t \ge 0$ ,

$$h(t) \doteq \min_{1 \le i \le d} c_i(\phi^*(t))_i,$$
  

$$I(t) \doteq \operatorname{argmin}_{1 \le i \le d} c_i(\phi^*(t))_i.$$

Obviously h is absolutely continuous. We claim that there exists a constant a > 0 such that

$$\dot{h}(t) \ge a \tag{A.12}$$

for almost every t. We first assume that  $I(t) \neq \{1, \ldots, d\}$ . It follows that  $I(t) \cap \pi(\phi^*(t)) = \emptyset$ . Therefore for any  $j \in I(t)$  (A.11) implies that

$$(\dot{\phi}^*(t))_j = \lambda_j e^{-\alpha_j^{A^*}}$$

Note that there exists a small neighborhood of t, say U, such that  $I(s) \subset I(t)$  for every  $s \in U$ . The claim (A.12) follows readily. It remains to show for the case where  $I(t) = \{1, \ldots, d\}$ . In this case,  $I(t) = \pi(\phi^*(t)) = \{1, \ldots, d\}$ . Since the Lebesgue measure of the set

$$\left\{t \ge 0: \pi(\phi^*(t)) = \{1, \dots, d\}, \pi(\dot{\phi}^*(t)) \neq \{1, \dots, d\}\right\}$$

is zero [4, Theorem A.6.3], we can further assume that  $\pi(\dot{\phi}^*(t)) = \{1, \ldots, d\}$ . In other words, there a exist constant b such that

$$b = c_i(\dot{\phi}^*(t))_i, \quad i = 1, ..., d.$$

Thanks to (A.11) and (4.5),

$$\frac{b}{c_j} = (\dot{\phi}^*(t))_j = \begin{cases} \frac{\lambda_j \mu_j}{\mu_j - z^{A^*}} - \rho_j^*(t)(\mu_j - z^{A^*}) & \text{if } j \in A^*, \\ \lambda_j - \rho_j^*(t)\mu_j & \text{if } j \notin A^*. \end{cases}$$

Since the summation of  $\{\rho_j^*(t) : j = 1, ..., d\}$  is one, it is not difficult to solve for b to obtain that

$$b = \left[\sum_{j \notin A^*} \frac{1}{c_j \mu_j} + \sum_{j \in A^*} \frac{1}{c_j (\mu_j - z^{A^*})}\right]^{-1} \cdot \left[\sum_{j \notin A^*} \frac{\lambda_j}{\mu_j} + \sum_{j \in A^*} \frac{\lambda_j \mu_j}{(\mu_j - z^{A^*})^2} - 1\right].$$

However, by the definition of  $z^{A^*}$  of (4.4),

$$\sum_{j \in A^*} \frac{\lambda_j \mu_j}{(\mu_j - z^{A^*})^2} - 1 = \sum_{j \in A^*} \frac{\lambda_j z^{A^*}}{\mu_j - z^{A^*}} > 0.$$

It follows that b > 0 and the claim (A.12) again holds.

The inequality (A.12) implies that the trajectory  $\phi^*$  stays in the positive orthant  $\mathbb{R}^d_+$  and the corresponding exit time  $\tau^*$  is finite. It remains to show the equality (A.7). Note that by (4.2), (A.9), and (A.10),

$$L(\phi^*(t), \dot{\phi}^*(t)) \le \sum_{i \in \pi(\phi^*(t))} \rho_i^*(t) L^{(i)}(DH^{(i)}(-\alpha^{A^*})).$$

By Lemma 4.1 and the conjugacy of  $-\alpha^{A^*}$  and  $DH^{(i)}(-\alpha^{A^*})$ ,

$$L^{(i)}(DH^{(i)}(-\alpha^{A^*})) = \langle -\alpha^{A^*}, DH^{(i)}(-\alpha^{A^*}) \rangle - H^{(i)}(-\alpha^{A^*}) \\ \leq \langle -\alpha^{A^*}, DH^{(i)}(-\alpha^{A^*}) \rangle$$

Therefore

$$L(\phi^*(t), \dot{\phi}^*(t)) \le \langle -\alpha^{A^*}, \dot{\phi}^*(t) \rangle.$$

Integrating both sides from 0 to  $\tau^*$  and using  $\phi^*(0) = 0$ , we have

$$\int_0^{\tau^*} L(\phi^*(t), \dot{\phi}^*(t)) dt \le \langle -\alpha^{A^*}, \phi^*(\tau^*) \rangle \le \langle -\alpha^{A^*}, \bar{c} \rangle = \gamma.$$

Since the inequality (A.6) holds for every  $\phi$ , in particular for  $\phi^*$ , the equality (A.7) follows readily.

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