# Optimal Stopping with Forced Exits

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#### Abstract

We consider a continuous time optimal stopping problem with multiple entries and forced exits. The value for such an optimization problem with a general payoff function is solved in closed form under the assumption that the state process is a geometric Brownian motion and the forced exits come in according to a Poisson process. The effect due to the forced exits is analyzed. It is shown that the presence of the forced exits is a true risk (meaning that it will reduce the value and enlarge the "continuation" region) if and only if the entry cost is large enough compared to the running cost.

## 1 Introduction

Classical stochastic optimal stopping problems usually assume that the decision maker has the right to select an entry time and/or an exit time according to the information history, in order to maximize or minimize a given functional. Such a formulation has been extensively studied and successfully applied to many disciplines including economics investment theory, financial option pricing, and so forth [1, 2, 3, 6, 9, 12, 13, 14]. The purpose of this paper is to study an alternative setup with multiple entries and *forced* exits. Also of interest is the effect on the optimal behavior due to the presence of these forced exits.

Consider the following standard formulation of an optimal stopping problem. Let c and k be two non-negative constants,  $X = (X_t)$  a non-negative

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stochastic process, and  $h : \mathbb{R}^+ \to \mathbb{R}^+$  a measurable function. The objective is to find an optimal stopping time (i.e., entry time) so as to maximize the total discounted payoff

$$\mathbb{E}\left[\int_{\tau}^{\infty} e^{-rt} \left(h(X_t) - c\right) \, dt - e^{-r\tau} k\right] \tag{1.1}$$

over all stopping times  $\tau$  taking values in  $[0, +\infty]$ . This optimization criterion is motivated by the classical economic investment valuation problem as follows. Suppose an investor must decide a (random) time  $\tau$  to invest in an economic project. The cost for initiating the project is k, while the operation of the project entails a running cost with constant rate c. The project will yield a payoff with a rate in the form of h(X) where X is some exogenous price process. For more details, please see [6], where a special case is studied with h(x) = x and X a geometric Brownian motion.

In this paper we consider the following optimal stopping problem. Let  $\{T_j\}$  be a given sequence of increasing, positive random variables, which stand for the (potential) times of forced exits. The objective is to judiciously choose a sequence of stopping times  $\{\tau_i\}$  so as to maximize

$$\mathbb{E}\sum_{i=1}^{\infty} \left[ \int_{\tau_i}^{\sigma_i} e^{-rt} (h(X_t) - c) \, dt - e^{-r\tau_i} k \right]$$

with the constraint  $\tau_n < \sigma_n \leq \tau_{n+1}$ , where  $\sigma_n \doteq \inf \{T_j : \tau_n < T_j\}$ , for every *n*. In other words,  $\tau_n$  indicates the *n*-th entry time, while  $\sigma_n$ , the (potential) exit time right after  $\tau_n$ , is the *n*-th forced exit time.

Formulations of this type do not seem to have attracted much attention in the probability literature. The present paper is concerned with the mathematical analysis of such models, and how the presence of these forced exits affects the optimal behavior as compared with that of the standard model with payoff (1.1). This model may also yield some interesting applications. For example, it may be used as an extension to the previously mentioned classical investment model, in order to incorporate the so-called liquidation risk [4]: Suppose the investor borrows money from a bank in order to finance the project. In return, the bank can be given the right to seize the project asset and put them to alternative uses (i.e., liquidation), at times it sees fit. The reason for such liquidation could be that the bank needs money for a new business or for consumption. Once liquidation happens, the investor will not be able to acquire the output from the project. However, he can restart the whole investment process again by finding another financier. In this case,  $\{T_i\}$  will be the sequence of potential liquidation times. **Remark 1.1** In our formulation, the exit times  $\{\sigma_n\}$  are completely determined by our choice of  $\{\tau_n\}$  and the pre-given sequence  $\{T_j\}$ . The case where the exit times  $\{\sigma_n\}$  are also chosen by the decision maker has been considered by many authors; see [5, 6, 7] and references therein. The latter will yield a larger value function than that from the standard model (1.1). This, however, is not always the case for our formulation; see Section 6 for more details.

**Remark 1.2** Of course, a fully fledged investment model with liquidation risk should be much more complicated. For example, we will assume  $\{T_j\}$ is independent of the price process X, and the entry cost k is fixed for each round of investment. More realistically, one should allow some correlations between  $\{T_j\}$  and X, and allow k to change for each investment cycle. But by making these simplifications, the model is more analytically tractable, and will still provide some insights on the effect of liquidation risk.

The paper is organized as follows. Section 2 gives the mathematical formulation of the optimization problem. In Section 3, we provide a brief review of an importance class of ODEs that will be very useful to our analysis. The variational inequality associated with the value function is presented in Section 4, which is then explicitly solved. A verification argument is given in Section 5. The effect of forced exits is analyzed in Section 6. It is discovered that the forced exit is a true risk (meaning that it will reduce the value function and enlarge the "continuation" region) if and only if the entry k is large compared to the running cost c. To ease exposition, some of the more technical proofs are deferred to the Appendix.

### 2 Model formulation

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space with filtration  $\mathbb{F} = (\mathcal{F}_t)$  satisfying the usual conditions: right-continuity and completion by *P*-negligible sets. It is assumed to be rich enough to carry a standard Brownian motion  $W = (W_t, \mathcal{F}_t)$  and a Poisson process  $N = (N_t, \mathcal{F}_t)$  with rate  $\lambda$ .

Let X be the geometric Brownian motion defined by

$$dX_t = bX_t dt + \sigma X_t dW_t, \quad X_0 = x, \tag{2.2}$$

where  $b, \sigma$  are constants with  $\sigma > 0$ , and  $x \in \mathbb{R}^+$  is the initial condition. The infinitisimal generator of X is denoted by  $\mathbb{L}$ ; i.e., for any twice continuously differentiable function  $f : \mathbb{R} \to \mathbb{R}$ ,

$$(\mathbb{L}f)(x) \doteq \frac{1}{2}\sigma^2 x^2 f''(x) + bxf'(x).$$
(2.3)

Let  $\{T_j\}$  denote the arrival times of the Poisson process N with convention  $T_0 = 0$ . Also let  $h : \mathbb{R}^+ \to \mathbb{R}^+$  be a measurable function, c, k two non-negative constants, and r > 0 the discount factor. The objective is to maximize the expected total discounted payoff

$$V(x) \doteq \sup_{\tau_i} \mathbb{E}^x \sum_{i=1}^{\infty} \left[ \int_{\tau_i}^{\sigma_i} e^{-rt} \left( h(X_t) - c \right) dt - e^{-r\tau_i} k \right]$$
(2.4)

over all sequence of stopping times  $\{\tau_i\}$  taking values in  $[0, \infty]$  with the constraint  $\tau_n < \sigma_n \leq \tau_{n+1}$  for every n, where  $\sigma_n \doteq \inf \{T_j : \tau_n < T_j\}$ .

Condition 2.1 We will make the following assumptions in the paper.

- 1. The Brownian motion W and the Poisson process N are independent.
- 2. The function h is continuous, non-decreasing, non-constant, and satisfies h(0) = 0. Furthermore, for every initial condition x,

$$\mathbb{E}^x \int_0^\infty e^{-rt} h(X_t) \, dt < \infty.$$

3. At least one of the constants c and k are strictly positive.

The inpendence of W and N is imposed for the convenience of analysis. Using the methodology of this paper, one may still be able to solve the more general case where the arrival rate of N is a function of W, but the analysis will become more cumbersome. Condition 2.1.2 is not difficult to verify, and it guarantees that the optimization problem (2.4) is well defined; see Remarks 2.1 and 3.2. Condition 2.1.3 is imposed to avoid the triviality.

**Remark 2.1** For every stopping time sequence  $\{\tau_i\}$ , we have

$$\sum_{i=1}^{\infty} \left| \int_{\tau_i}^{\sigma_i} e^{-rt} (h(X_t) - c) \, dt - e^{-r\tau_i} k \right| \le \int_0^{\infty} e^{-rt} (h(X_t) + c) \, dt + \sum_{i=1}^{\infty} e^{-r\tau_i} k dt$$

However, it is not difficult to see that  $\tau_i \geq T_{i-1}$  for all  $i = 1, 2, \ldots$  Therefore,

$$\mathbb{E}^{x} \sum_{i=1}^{\infty} e^{-r\tau_{i}} k \le k \mathbb{E}^{x} \sum_{i=1}^{\infty} e^{-rT_{i-1}} = k \mathbb{E}^{x} \cdot \sum_{i=1}^{\infty} (\lambda/(\lambda+r))^{i-1} = k(\lambda+r)/r,$$

which in turn implies that the optimization problem (2.4) is well defined.

## 3 Review of an important ODE

In this section, we discuss a class of ordinary differential equations that will play an important role in the analysis.

Recall the definition (2.3) and consider the second order ODE of form

$$-(r+\lambda)f + \mathbb{L}f + h = 0. \tag{3.5}$$

A pair of fundamental solutions associated with this equation is  $(x^{\beta_+(\lambda)}, x^{\beta_-(\lambda)})$ , where

$$\beta_{\pm}(\lambda) \doteq \left(\frac{1}{2} - \frac{b}{\sigma^2}\right) \pm \sqrt{\left(\frac{1}{2} - \frac{b}{\sigma^2}\right)^2 + \frac{2(\lambda+r)}{\sigma^2}}.$$
 (3.6)

It is not difficult to see that, for every non-negative  $\lambda$ ,

$$\beta_{-}(\lambda) < 0 < \beta_{+}(\lambda). \tag{3.7}$$

**Remark 3.1** To ease exposition, we will use  $\beta_{\pm}$  instead of  $\beta_{\pm}(\lambda)$  when no confusion is incurred. We will also write  $\alpha_{\pm} \doteq \beta_{\pm}(0)$ .

**Proposition 3.1** Under Condition 2.1.2, the ODE (3.5) admits a twice continuously differentiable particular solution on  $\mathbb{R}^+$ 

$$\begin{aligned} H_{\lambda}(x) &\doteq \frac{2}{\sigma^{2}(\beta_{+}-\beta_{-})} \left[ x^{\beta_{-}} \int_{0}^{x} s^{-\beta_{-}-1} h(s) \, ds + x^{\beta_{+}} \int_{x}^{\infty} s^{-\beta_{+}-1} h(s) \, ds \right] \\ &= \frac{2}{\sigma^{2}(\beta_{+}-\beta_{-})} \left[ \int_{0}^{1} s^{-\beta_{-}-1} h(xs) \, ds + \int_{1}^{\infty} s^{-\beta_{+}-1} h(xs) \, ds \right]. \end{aligned}$$

Furthermore, the function  $H_{\lambda}$  admits the stochastic representation

$$H_{\lambda}(x) = \mathbb{E}^x \int_0^\infty e^{-(r+\lambda)t} h(X_t) \, dt.$$

**Proof.** The proof can be found in [10, 11].

**Remark 3.2** The paper [10] also proved the following result, which is very useful for the verification of Condition 2.1.2. Given  $\lambda \geq 0$  and any non-negative function h, if the function  $H_{\lambda}(x)$  as defined in Proposition 3.1 is finite for every x, then

$$\mathbb{E}^x \int_0^\infty e^{-(r+\lambda)t} h(X_t) \, dt < \infty.$$

The next result is also useful to the analysis. Its proof is deferred to Appendix A.

**Lemma 3.2** Assume that Condition 2.1.2 holds. Then for any  $\lambda \geq 0$ 

$$\liminf_{x \to \infty} x^{-\beta_+} H_\lambda(x) = 0.$$

Given any initial condition  $X_0 \equiv x > 0$ , we have, with probability one,

$$\lim_{t \to \infty} e^{-(r+\lambda)t} H_{\lambda}(X_t) = 0.$$

Moreover, for any stopping time sequence  $\{\tau_n\}$  such that  $\tau_n \to \infty$  almost surely, we have

$$\lim_{n \to \infty} \mathbb{E}^x \left[ e^{-(r+\lambda)\tau_n} H_\lambda(X_{\tau_n}) \right] = 0.$$

# 4 An auxiliary optimization problem and the variational inequality

To facilitate the analysis, we will introduce an auxiliary optimization problem as follows. Recall that  $\{T_j\}$  is the arrival times of the Poisson process N. Define the value function

$$\bar{V}(x) \doteq \sup_{\tau_i} \mathbb{E}^x \left\{ \int_0^{T_1} e^{-rt} (h(X_t) - c) dt + \sum_{i=1}^{\infty} \left[ \int_{\tau_i}^{\sigma_i} e^{-rt} (h(X_t) - c) dt - e^{-r\tau_i} k \right] \right\},$$
(4.8)

where the supremum is over all stopping time sequences  $\{\tau_i\}$  such that  $T_1 \leq \tau_1$ , and  $\tau_n < \sigma_n \leq \tau_{n+1}$  for every *n* where  $\sigma_n \doteq \inf \{T_j : \tau_n < T_j\}$ .

If we compare this optimization problem with the definition (2.4) of V, the only different piece is the extra first integral in (4.8). One can interpret V and  $\bar{V}$  as follows. At any time, the decision maker is either waiting to make an entry (say, "idle"), or has already made one but not been forced out yet (say, "active"). The value function V is the total expected discounted payoff the decision maker can achieve if he is "idle", while  $\bar{V}$  if the decision maker is "active".

#### 4.1 Heuristics for the variational inequality

We will proceed heuristically in order to derive the variational inequality that the pair  $(V, \bar{V})$  should satisfy. It is not unnatural to guess that the optimal policy for an "idle" decision maker should take the following form: make an entry whenever the process X exceeds some threshold  $x^*$ , and wait otherwise. As for an "active" decision maker, there is no choice until the first forced exit time  $T_1$ , after which the decision maker becomes "idle" and it should follow the same strategy described above to behave optimally.

If it is optimal for an "idle" decision maker to wait when the process X is below  $x^*$ , then one would expect

$$-rV + \mathbb{L}V = 0$$
, for every  $x \in (0, x^*)$ .

However, when the process X exceeds  $x^*$ , the "idle" decision maker will pay entry cost k and become "active". Therefore, one expects that

$$V = \overline{V} - k$$
, for every  $x \ge x^*$ .

Now let us see what equation that  $\overline{V}$  satisfies. Consider an "active" decision maker with the price state  $X_0 = x$ . In a small time interval of length  $\Delta t$ , the Poisson process N has probability  $\lambda \Delta t$  to make a jump; i.e., with probability  $\lambda \Delta t$  the decision maker will be forced to exit and become "idle" and with probability  $1 - \lambda \Delta t$ , the decision maker will stay "active". If the decision maker behave optimally afterwards, he will achieve the value  $V(X_{\Delta t})$ . This suggests, at least formally, that

$$\begin{split} \bar{V}(x) &= (h(x) - c) \triangle t + \lambda \triangle t \cdot V(X_{\triangle t}) + (1 - \lambda \triangle t) e^{-r \triangle t} \mathbb{E}^x \, \bar{V}(X_{\triangle t}) \\ &\approx (h(x) - c) \triangle t + \lambda \triangle t \cdot V(x) + (1 - \lambda \triangle t) [\bar{V}(x) + (-r\bar{V} + \mathbb{L}\bar{V}) \triangle t] \\ &= \bar{V}(x) + [-r\bar{V} + \mathbb{L}\bar{V} + \lambda(V - \bar{V}) + h(x) - c] \triangle t, \end{split}$$

which yields

$$-r\bar{V} + \mathbb{L}\bar{V} + \lambda(V - \bar{V}) + h(x) - c = 0, \text{ for every } x > 0.$$

We obtain the following variational inequality from the preceding heuristic argument.

Variational Inequality: Find a pair of functions  $v \in \mathcal{C}^1(\mathbb{R}^+)$ ,  $v_1 \in \mathcal{C}^2(\mathbb{R}^+)$ , and a constant  $x^* > 0$ , such that

$$-rv + \mathbb{L}v = 0;$$
 if  $0 < x < x^*$ , (4.9)

$$v - (\bar{v} - k) = 0;$$
 if  $x \ge x^*$ , (4.10)

$$-r\bar{v} + \mathbb{L}\bar{v} + \lambda(v - \bar{v}) + h(x) - c = 0; \quad \text{if } x > 0 \quad (4.11)$$

hold, and the growth condition

$$|v(x)| \le \varepsilon_1 H_0(x) + \varepsilon_2, \quad |\bar{v}(x)| \le \varepsilon_1 H_0(x) + \varepsilon_2 \tag{4.12}$$

is satisfied for some positive constants  $\varepsilon_1, \varepsilon_2$  and every x > 0.

**Remark 4.1** The value functions V and  $\overline{V}$  easily satisfy the growth condition (4.12). For example, it is clear from the definition (2.4) and Proposition 3.1 that

$$0 \le V(x) \le \mathbb{E}^x \int_0^\infty e^{-rt} h(X_t) \, dt = H_0(x),$$

The growth condition for  $\overline{V}$  is similarly obtained.

**Remark 4.2** Thanks to (4.10) and (4.11), v satisfies

$$-rv + \mathbb{L}v - (r+\lambda)k + h(x) - c = 0, \quad \forall \ x > x^*.$$
(4.13)

However, equations (4.9), (4.13), the smooth-fit-principle, and the growth condition alone are not enough to yield a unique solution for v. On the other hand, the addition of function  $v_1$  does ensure a unique solution to the variational inequality; see Proposition 4.1. This says that the introduction of the auxiliary problem (4.8) is somewhat necessary.

#### 4.2 The solution to the variational inequality

The task of this section is to explicitly compute the solution to the variational inequality.

Recall  $\alpha_{\pm}$  as defined in Remark 3.1. It follows from equation (4.9) that, for  $x < x^*$ ,

$$v(x) = A_+ x^{\alpha_+} + A_- x^{\alpha_-},$$

for some constants  $A_{\pm}$ . However, the growth condition (4.12) ensures that v is bounded around the neighborhood of 0, which in turns implies that  $A_{-} = 0$  since  $\alpha_{-} < 0$ . Therefore,

$$v(x) = A_+ x^{\alpha_+}, \quad \text{if } x \in (0, x^*).$$
 (4.14)

On the other hand, equations (4.9) and (4.11) yield

 $-(r+\lambda)(v-\bar{v}) + \mathbb{L}(v-\bar{v}) + h(x) - c = 0, \quad \text{for } x \in (0, x^*).$ 

Thanks to the results in Section 3, we have

$$v - \bar{v} = B_+ x^{\beta_+} + B_- x^{\beta_-} - H_\lambda(x) + c/(r+\lambda), \quad \text{ if } \ x \in (0, x^*),$$

for some constants  $B_{\pm}$ . Similarly, the growth condition (4.12) implies  $B_{-} = 0$ . Thus we have

$$\bar{v}(x) = A_+ x^{\alpha_+} - B_+ x^{\beta_+} + H_\lambda(x) - c/(r+\lambda), \quad \text{if } x \in (0, x^*).$$
(4.15)

On the interval  $(x^*, \infty)$ , equations (4.10) and (4.11) imply that

$$-r\bar{v} + \mathbb{L}\bar{v} + h(x) - c - \lambda k = 0$$

Again, thanks to the results in Section 3, we have (abusing the notation)

$$\bar{v}(x) = Cx^{\alpha_+} + A_- x^{\alpha_-} + H_0(x) - (c + \lambda k)/r, \text{ if } x \in (x^*, \infty),$$

for some constants C and  $A_-$ . However, it is clear that C = 0 because of the growth condition (4.12) and that  $\liminf_{x\to\infty} x^{-\alpha_+} H_0(x) = 0$  (see Lemma 3.2). Hence

$$\bar{v}(x) = A_{-}x^{\alpha_{-}} + H_{0}(x) - (c + \lambda k)/r, \quad \text{if } x \in (x^{*}, \infty),$$
 (4.16)

and

$$v(x) = A_{-}x^{\alpha_{-}} + H_{0}(x) - [c + (r + \lambda)k]/r, \quad \text{if} \ x \in (x^{*}, \infty).$$
(4.17)

It remains to determine the four unknowns  $(x^*; A_+, A_-, B_+)$ . However, the continuity of  $v, v', \bar{v}, \bar{v}'$  across the optimal investment boundary  $x^*$  yields four equations as follows.

$$\begin{aligned} A_{-}(x^{*})^{\alpha_{-}} + H_{0}(x^{*}) &- [c + (r + \lambda)k]/r &= A_{+}(x^{*})^{\alpha_{+}}, \\ \alpha_{-}A_{-}(x^{*})^{\alpha_{-}} + x^{*}H_{0}'(x^{*}) &= \alpha_{+}A_{+}(x^{*})^{\alpha_{+}}, \\ -B_{+}(x^{*})^{\beta_{+}} + H_{\lambda}(x^{*}) - c/(r + \lambda) &= k, \\ -\beta_{+}B_{+}(x^{*})^{\beta_{+}} + x^{*}H_{\lambda}'(x^{*}) &= 0. \end{aligned}$$

$$(4.18)$$

Some algebra yields that  $x^*$  is the solution to the equation

$$\int_{0}^{1} s^{-\beta_{-}-1} h(xs) \, ds = \frac{\sigma^{2} \beta^{+}}{2} \left( \frac{c}{r+\lambda} + k \right) \tag{4.19}$$

and

$$A_{\pm} = \frac{(x^*)^{-\alpha_{\pm}}}{\alpha_{+} - \alpha_{-}} \cdot \left[ x^* H_0'(x^*) - \alpha_{\mp} H_0(x^*) + \alpha_{\mp} \frac{c + (r + \lambda)k}{r} \right], \qquad (4.20)$$

$$B_{+} = (x^{*})^{-\beta_{+}} \cdot x^{*} H_{\lambda}'(x^{*}) / \beta_{+}.$$
(4.21)

We have the following proposition, whose proof is technical and is deferred to Appendix. **Proposition 4.1** Assume that Condition 2.1.2 and Condition 2.1.3 hold. Let  $h(\infty) \doteq \lim_{x\to\infty} h(x)$ , and assume  $h(\infty) > c + (r + \lambda)k$ . Then equation (4.19) admits a unique, strictly positive solution  $x^*$ . The constants  $(A_+, A_-, B_+)$  defined by (4.20) and (4.21) are all strictly positive. Furthermore, the triple  $(v, \bar{v}; x^*)$  determined by equations (4.14)–(4.17) is the unique solution to the variation inequality (4.9)–(4.12). In particular, the function v is non-negative, non-decreasing, and satisfy

$$v(x) \ge \bar{v}(x) - k,$$

for all x > 0, where the equality holds if and only if  $x \ge x^*$ .

### 5 The solution to the optimization problems

It is expected that the solution v to the variational inequality is the value function of the optimization problem (2.4), and  $x^*$  is the optimal threshold. In this section we give a rigorous proof of this result.

**Theorem 5.1** Assume Condition 2.1, and let  $h(\infty) \doteq \lim_{x\to\infty} h(x)$ .

- 1. If  $h(\infty) \leq c + (r+\lambda)k$ , then  $V(x) \equiv 0$ , and it is optimal never to make an entry, or  $\tau_n^* \equiv \infty$  for every n.
- Assume h(∞) > c + (r + λ)k. Let (v, v̄; x\*) be the unique solution the variational inequality (4.9)-(4.12). Then V(x) = v(x) for every x > 0. Furthermore, it is optimal to make an entry whenever the state process X exceeds the threshold x\*. In other words, the sequence of optimal entry times is recursively defined by

$$\tau_n^* \doteq \inf \left\{ t \ge \sigma_{n-1}^* : X_t \ge x^* \right\}$$

for every  $n \ge 1$ , where  $\sigma_0^* \doteq 0$  and  $\sigma_n^* \doteq \inf \{T_j : \tau_n^* < T_j\}$  for  $n \ge 1$ .

**Proof.** In this proof, for any stopping time  $\tau$ , the values of  $e^{-r\tau}v(X_{\tau})$  and  $e^{-r\tau}\overline{v}(X_{\tau})$  on the set  $\{\tau = \infty\}$  are naturally set as 0, thanks to the growth condition (4.12) and Lemma 3.2.

We will first give the proof for the second case. Assume that  $h(\infty) > c + (r + \lambda)k$ . We will start by showing that, for any stopping time  $\tau$  and  $\sigma \doteq \inf\{T_j : \tau < T_j\}$  the first Poisson arrival time after  $\tau$ , we have

$$e^{-r\tau}\bar{v}(X_{\tau}) = \mathbb{E}^{x}\left[\int_{\tau}^{\sigma} e^{-rt}(h(X_{t}) - c) dt + e^{-r\sigma}v(X_{\sigma}) \middle| \mathcal{F}_{\tau}\right].$$
 (5.22)

Thanks to strong Markov property of process X and the memoryless property of exponential random variables, it is sufficient to show

$$\bar{v}(x) = \mathbb{E}^x \left[ \int_0^U e^{-rt} (h(X_t) - c) \, dt + e^{-rU} v(X_U) \right], \quad (5.23)$$

where U is an independent exponential random variable with rate  $\lambda$ . Consider the process  $M = (M_t, \mathcal{F}_t)$  where

$$M_t \doteq e^{-(r+\lambda)t}\bar{v}(X_t) + \int_0^t e^{-(r+\lambda)s} \left[h(X_s) - c + \lambda v(X_s)\right] \, ds.$$

Since  $\bar{v}$  is twice continuously differentiable, we can apply Itô formula and equation (4.11) to obtain

$$M_t = \bar{v}(x) + \int_0^t e^{-(r+\lambda)s} \bar{v}'(X_s) \sigma X_s \, dW_s.$$

However, it is not difficult to see  $|x\bar{v}'(x)| \leq c_1 x^{\alpha_+} + c_2$  for some constants  $c_1, c_2$ , thanks to equations (4.15), (4.16), and (A.3). Hence the above stochastic integral, or the process M, defines a true martingale. Therefore, for every  $t \geq 0$ , we have

$$\bar{v}(x) = \mathbb{E}^x e^{-(r+\lambda)t} \bar{v}(X_t) + \mathbb{E}^x \int_0^t e^{-(r+\lambda)s} [h(X_s) - c + \lambda v(X_s)] \, ds. \quad (5.24)$$

But the growth condition (4.12) and Proposition 3.1 imply that

$$\lim_{t \to \infty} \mathbb{E}^x e^{-rt} |\bar{v}(X_t)| = 0.$$

Furthermore, the Monotone Convergence Theorem yields

$$\lim_{t \to \infty} \mathbb{E}^x \int_0^t e^{-(r+\lambda)s} [h(X_s) - c + \lambda v(X_s)] \, ds$$
$$= \mathbb{E}^x \int_0^\infty e^{-(r+\lambda)s} [h(X_s) - c + \lambda v(X_s)] \, ds$$

However, note that

$$\mathbb{E}^x \int_0^\infty e^{-(r+\lambda)s} \lambda v(X_s) \, ds = \mathbb{E}^x \int_0^\infty e^{-rs} v(X_s) \cdot \lambda e^{-\lambda s} \, ds = \mathbb{E}^x e^{-rU} v(X_U)$$

and by Fubini's Theorem

$$\mathbb{E}^{x} \int_{0}^{\infty} e^{-(r+\lambda)s} (h(X_{s}) - c) \, ds \qquad (5.25)$$

$$= \mathbb{E}^{x} \int_{0}^{\infty} e^{-rs} (h(X_{s}) - c) \int_{s}^{\infty} \lambda e^{-\lambda t} \, dt \, ds$$

$$= \mathbb{E}^{x} \int_{0}^{\infty} \left[ \int_{0}^{t} e^{-rs} (h(X_{s}) - c) \, ds \right] \cdot \lambda e^{-\lambda t} \, dt$$

$$= \mathbb{E}^{x} \int_{0}^{U} e^{-rs} (h(X_{s}) - c) \, ds.$$

Now letting  $t \to \infty$  in equation (5.24), we arrive at (5.23), which yields (5.22) readily.

We now show that the process  $\{e^{-rt}v(X_t)\}$  is a non-negative supermartingale. The non-negativity is trivial from Proposition 4.1. Even though v is only continuously differentiable, one can still apply Itô formula [8, Exercise 6.24], and it follows that

$$Y_t \doteq e^{-rt} v(X_t) - \int_0^t [-rv(X_s) + \mathbb{L}v(X_s)] \, ds$$
 (5.26)

is a local martingale. However, by (4.9), (4.13), (A.6), we have  $-rv + \mathbb{L}v \leq 0$ . Therefore, Y is non-negative, whence a supermartingale. It is now easy to see that  $\{e^{-rt}v(X_t)\}$  is also a supermartingale.

Fix an arbitrary stopping times  $\tau$ . Let  $\sigma$  is the first passage time of level  $x^*$  after  $\tau$ ; i.e.,

$$\sigma \doteq \inf \left\{ t \ge \tau : X_t \ge x^* \right\}.$$

We claim that

$$e^{-r\tau}v(X_{\tau}) = \mathbb{E}^{x}\left[e^{-r\sigma}v(X_{\sigma})\middle|\mathcal{F}_{\tau}\right], \qquad (5.27)$$

Thanks to strong Markov property, we can assume  $\tau = 0$  without loss of generality. It follows from (5.26) and (4.9) that

$$Y_{t\wedge\sigma} = e^{-r(t\wedge\sigma)}v(X_{t\wedge\sigma}) - \int_0^{t\wedge\sigma} (-rv + \mathbb{L}v)(X_u) \, du = e^{-r(t\wedge\sigma)}v(X_{t\wedge\sigma}),$$

which is a local martingale. Clearly,  $\left\{e^{-r(t\wedge\sigma)}v(X_{t\wedge\sigma})\right\}$  is bounded, hence it is a true martingale. Therefore,

$$v(x) = \mathbb{E}^x \left[ e^{-r(t \wedge \sigma)} v(X_{t \wedge \sigma}) \right],$$

for every t. Letting  $t \to \infty$ , (5.27) (with  $\tau = 0$ ) follows readily from Bounded Convergence Theorem.

We are now in a position to prove that V(x) = v(x) and that  $\{\tau_n^*\}$  define an optimal sequence of entry times. Fix an arbitrary sequence of stopping times  $\{\tau_n\}$  such that  $\tau_n < \sigma_n \leq \tau_{n+1}$ , where  $\sigma_n \doteq \inf\{T_j : \tau_n < T_j\}$ . We have

$$\begin{aligned}
v(x) &\geq \mathbb{E}^{x} \left[ e^{-r\tau_{1}} v(X_{\tau_{1}}) \right] \\
&\geq \mathbb{E}^{x} \left[ e^{-r\tau_{1}} \bar{v}(X_{\tau_{1}}) - e^{-r\tau_{1}} k \right] \\
&= \mathbb{E}^{x} \left[ \int_{\tau_{1}}^{\sigma_{1}} e^{-rt} (h(X_{t}) - c) \, dt + e^{-r\sigma_{1}} v(X_{\sigma_{1}}) - e^{-r\tau_{1}} k \right] \\
&\geq \mathbb{E}^{x} \left[ \int_{\tau_{1}}^{\sigma_{1}} e^{-rt} (h(X_{t}) - c) \, dt - e^{-r\tau_{1}} k \right] + \mathbb{E}^{x} e^{-r\tau_{2}} v(X_{\tau_{2}}).
\end{aligned}$$

Here the first and the last inequalities hold since  $\{e^{-rt}v(X_t)\}$  is a nonnegative supermartingale; the second inequality follows from Proposition 4.1; the equality is due to (5.22). Repeating the above operation, it follows that

$$\begin{aligned} v(x) &\geq & \mathbb{E}^{x} \sum_{i=1}^{n} \left[ \int_{\tau_{i}}^{\sigma_{i}} e^{-rt} (h(X_{t}) - c) \, dt - e^{-r\tau_{i}} k \right] + \mathbb{E}^{x} e^{-r\tau_{n+1}} v(X_{\tau_{n+1}}) \\ &\geq & \mathbb{E}^{x} \sum_{i=1}^{n} \left[ \int_{\tau_{i}}^{\sigma_{i}} e^{-rt} (h(X_{t}) - c) \, dt - e^{-r\tau_{i}} k \right] \end{aligned}$$

for all  $n \ge 1$ . Letting  $n \to \infty$ , applying the Dominated Convergence Theorem (thanks to Remark 2.1), and then taking supremum over all possible sequence  $\{\tau_n\}$ , we have  $v(x) \ge V(x)$ .

However, it is easy to see, when  $\tau_i = \tau_i^*$  as defined in Theorem 5.1, all the inequalities above become equalities, thanks to Proposition 4.1 and (5.27). We have

$$v(x) = \mathbb{E}^{x} \sum_{i=1}^{n} \left[ \int_{\tau_{i}^{*}}^{\sigma_{i}^{*}} e^{-rt} (h(X_{t}) - c) dt - e^{-r\tau_{i}^{*}} k \right] + \mathbb{E}^{x} e^{-r\tau_{n+1}^{*}} v(X_{\tau_{n+1}^{*}})$$

for all  $n \ge 1$ . It is not difficult to see that  $\tau_n^* \ge T_{n-1}$ , which implies that  $\tau_n^* \to \infty$  almost surely. Thanks to Lemma 3.2 and the growth condition (4.12) we have

$$\lim_{n \to \infty} \mathbb{E}^{x} e^{-r\tau_{n+1}^*} v(X_{\tau_{n+1}^*}) = 0.$$

Letting  $n \to \infty$ , we arrive at

$$v(x) = \mathbb{E}^x \sum_{i=1}^{\infty} \left[ \int_{\tau_i^*}^{\sigma_i^*} e^{-rt} (h(X_t) - c) \, dt - e^{-r\tau_i^*} k \right]$$

by the Dominated Convergence Theorem. It follows that  $v(x) \leq V(x)$ , which in turns implies that v(x) = V(x) and  $\{\tau_n^*\}$  is the optimal strategy. We complete the proof for the second case.

Now assume that  $h(\infty) \leq c + (r + \lambda)k$ . It suffices to show that for every stopping time sequence  $\{\tau_n\}$ , we have

$$\mathbb{E}^x \left[ \int_{\tau_i}^{\sigma_i} e^{-rt} (h(X_t) - c) \, dt - e^{-r\tau_i} k \right] \le 0,$$

for every i. Again thanks to the strong Markov property of X and the memoryless property of exponential distributions, we only need to show

$$\mathbb{E}^{x}\left[\int_{0}^{U} e^{-rt}(h(X_{t})-c)\,dt\right] \leq k,$$

where U is an independent exponential random variable with rate  $\lambda$ . But equation (5.25) and the monotonicity of h imply that

$$\mathbb{E}^{x}\left[\int_{0}^{U} e^{-rt}(h(X_{t})-c) dt\right] = \mathbb{E}^{x} \int_{0}^{\infty} e^{-(r+\lambda)s}(h(X_{s})-c) ds$$
  
$$\leq (h(\infty)-c)/(r+\lambda)$$
  
$$\leq k.$$

This completes the proof.

**Remark 5.1** Under the conditions of Theorem 5.1, one can similarly prove that, for the auxiliary optimization problem (4.8), the value function  $\overline{V} = \overline{v}$  and the optimal entry threshold is  $x^*$ .

**Corollary 5.2** Assume that Condition 2.1 holds, and let  $h(\infty) \doteq \lim_{x\to\infty} h(x)$ . Consider the classical optimal stopping problem

$$V_0(x) \doteq \sup_{\tau} \mathbb{E}^x \left[ \int_{\tau}^{\infty} e^{-rt} (h(X_t) - c) \, dt - e^{-r\tau} k \right].$$

- 1. If  $h(\infty) \leq c + rk$ , then  $V_0(x) \equiv 0$ , and  $\tau^* \equiv \infty$  is optimal.
- 2. Assume  $h(\infty) > c+rk$ . The optimal exercise boundary  $x^*$  is the unique solution to the equation

$$\int_0^1 s^{-\alpha_- - 1} h(xs) \, ds = \frac{\sigma^2 \alpha_+}{2} \left(\frac{c}{r} + k\right)$$

and the value function is

$$V_0(x) = \left\{ egin{array}{ccc} A_+ x^{lpha_+} & ; & if \ x \leq x^* \ H_0(x) - [c+rk]/r & ; & if \ x > x^* \end{array} 
ight.$$

Here

$$A_{+} \doteq \frac{(x^{*})^{-\alpha_{+}}}{\alpha_{+} - \alpha_{-}} \cdot \left[ x^{*} H_{0}'(x^{*}) - \alpha_{-} H_{0}(x^{*}) + \alpha_{-} \frac{c + rk}{r} \right]$$

is a strictly positive constant.

**Proof.** The proof is similar and omitted.

**Remark 5.2** It follows from Theorem 5.1 and Corollary 5.2 that the optimal exercise threshold for the optimization problem (2.4) with forced exits is the same as that of the classical optimization problem with a discount factor  $r + \lambda$  instead of r. Note that the claim is not true for the value function.

## 6 Effects of forced exits

In this section, we study the effect on the optimal behavior due to the presence of the forced exits. We will slightly alter the notation. Instead of simply using  $x^*$  and V respectively for the optimal exercise boundary and the value function, we will write  $x^*(\lambda)$  and  $V(x; \lambda)$ , in order to distinguish among different Poisson arrival rates  $\lambda$ . The case  $\lambda = 0$  corresponds to the classical model without forced exits. We will adopt the following assumption:

Condition 6.1  $h(\infty) = \infty$ .

Even though the results in this section hold under weaker conditions, Condition 6.1 greatly eases the exposition.

The next lemma states that as  $\lambda \to 0$ , the optimal exercise boundary and the value function converge to those of the classical model, respectively.

Lemma 6.1 Assume that Conditions 2.1 and 6.1 hold. We have

$$\lim_{\lambda \to 0} x^*(\lambda) = x^*(0), \qquad \lim_{\lambda \to 0} V(x;\lambda) = V(x;0).$$

This convergence result is not surprising at all. The proof is trivial from Theorem 5.1 and Corollary 5.2, and thus omitted.

A question that arises quite naturally is as follows. Will the presence of the forced exits raises the optimal exercise boundary and reduces the

value function, compared to the classical model? If we consider our model as an economics investment model with liquidation risk, then this question is asking if the presence of liquidation risk will make the investment policy more conservative and reduce the value of the investment. For this reason, we will call the presence of forced exits a *true risk* if it raises the optimal exercise boundary and reduce the value function.

It is not difficult to guess that the answer depends very much on the relative size of the entry cost k to the running cost c. Consider the extreme case when k = 0. The presence of forced exits will only enlarge the value. Indeed, when a forced exit arrives, the decision maker can either immediately make an entry without any cost as if the forced exit had never occurred, or just choose to wait for better times to enter. Hence the forced exits only serve as a free opportunity to the decision maker. In contrast, if c = 0, then it is clear that the presence of forced exits always reduces the value.

It will be our main task in this section to give the necessary and sufficient condition for the presence of forced exits to be a true risk. More interestingly, we also give the necessary and sufficient condition so that the presence of forced exits is a true risk *no matter* what the value of the arrival rate  $\lambda$  is. Indeed, this is the case if and only if the entry cost k is large compared to the running cost c.

**Theorem 6.2** Assume that Conditions 2.1 and 6.1 hold. Suppose k > 0. Then the following conclusions hold.

- 1. There exists a cirtical threshold  $\bar{\lambda} \geq 0$ , such that  $x^*(\lambda) \geq x^*(0)$  and  $V(x;\lambda) \leq V(x;0)$  for all  $x \in \mathbb{R}^+$  if and only if  $\lambda \geq \bar{\lambda}$ .
- 2. Furthermore, the threshold  $\overline{\lambda} = 0$  if and only if  $k \ge g(c)$  for some continuous function  $g: \mathbb{R}^+ \to \mathbb{R}^+$  with g(0) = 0 and  $g(\infty) = \infty$ . The function g is sublinear in the sense that  $g(c) \le 2c/(\sigma\alpha_-)^2$ .
- 3. The function g is non-decreasing, if we further assume that  $h(x) \doteq h(e^x)$  is convex; e.g.,  $h(x) = x^p$  with p > 0.

**Remark 6.1** Theorem 6.2 asserts that the presence of forced exits is a true risk regardless of the arrival rate  $\lambda$  if and only if  $k \ge g(c)$ . It is also interesting to note that g is not always non-decreasing, as Example 2 shows.

The proof of Theorem 6.2 relies on the following lemma, which is interesting by itself. It asserts that the value is reduced whenever the optimal exercise boundary is raised. The converse, however, is not true, as shown by Example 1 below. **Lemma 6.3** Assume that Conditions 2.1 and 6.1 hold. For any  $\lambda$ , if  $x^*(\lambda) \ge x^*(0)$ , then  $V(x;\lambda) \le V(x;0)$  for every  $x \in \mathbb{R}^+$ .

The proof of this lemma and Theorem 6.2 is lengthy and technical, and will be deferred to Appendix B.

**Remark 6.2** For the case k = 0, it is straightforward to show that  $x^*(\lambda) \le x^*(0)$  and  $V(x;\lambda) \ge V(x;0)$  for every x.

### 7 Examples

**Example 1** We will consider the example with h(x) = x. It is not difficult to check that now Condition 2.1.2 is equivalent to b < r.

One can explicitly solve equation (4.19) to determine the optimal exercise boundary for every  $\lambda$ :

$$x^*(\lambda) = ch_1(\lambda) + kh_2(\lambda),$$

where the functions  $h_1, h_2$  are defined as

$$h_1(\lambda) \doteq \frac{\beta_+}{\beta_+ - 1} \frac{r - b + \lambda}{r + \lambda}, \qquad h_2(\lambda) \doteq \frac{\beta_+}{\beta_+ - 1} (r - b + \lambda).$$

With some algebra, one can verify that  $h'_1(0) < 0$  and  $h'_2(0) > 0$ . Let

$$\theta \doteq -h_1'(0)/h_2'(0).$$

We claim that g(c) is a linear function. More precisely, we have  $g(c) = \theta c$ . Indeed, observe that

$$\left. \frac{dx^*}{d\lambda} \right|_{\lambda=0} = ch_1'(0) + kh_2'(0) = h_2'(0)(k - \theta c).$$

If  $k > \theta c$ , then  $x^*(\lambda) > x^*(0)$  for  $\lambda$  small enough. Thanks to Theorem 6.2 and Lemma 6.3, we have  $\overline{\lambda} = 0$ , which in turn implies that  $g(c) \leq \theta c$ . On the other hand, if  $k < \theta c$ , then  $x^*(\lambda) < x^*(0)$  for  $\lambda$  small enough. Thus  $\overline{\lambda} > 0$ , and  $g(c) \geq \theta c$ . To conclude, we must have  $g(c) = \theta c$ .

Numerical Illustration: Set r = 0.2, b = 0.15 and  $\sigma = 1$ . It follows that  $\theta \approx 5.9$ . Let c = 1. Then  $\overline{\lambda} = 0$  if and only if  $\lambda \ge g(c) = \theta c \approx 5.9$ .

Figure 1 shows the difference of optimal exercise boundary  $x^*(\lambda) - x^*(0)$ as a function of  $\lambda$ , for k = 1, 3, 7. Clearly, for each k, the critical threshold  $\overline{\lambda}$  is the intersection of the corresponding curve with the horizontal dotted line. For instance,  $\bar{\lambda} \approx 1.5$  if k = 1, and  $\bar{\lambda} \approx 0.25$  if k = 3. When k = 7, as expected, we have  $\bar{\lambda} = 0$ .

Figure 2 displays the difference of value functions  $V(x; \lambda) - V(x; 0)$  as a function of x with the Poisson arrival rate  $\lambda$  set as 0.1. Note that the case k = 3 shows that the converse of Lemma 6.3 cannot be true in general.

**Example 2** This example is constructed in order to show that the function *g* need not be non-decreasing in general.

Let the parameters be set as  $b = 0, r = 1, \sigma = 1$ , and the payoff function

$$h(x) \doteq \left\{ egin{array}{cccc} x & ; & ext{if} \ \ 0 \leq x \leq 1 \ 1 & ; & ext{if} \ \ 1 < x \leq M \ x - (M-1) & ; & ext{if} \ \ x > M \end{array} 
ight.$$

Here M is large positive constant whose specific value is of little importance. Clearly, the function  $\bar{h}(x) \doteq h(e^x)$  is not convex.

It follows from the proof of Theorem 6.2 that  $\overline{\lambda} = 0$  if and only if

$$\phi(c,k)\doteq -\left.rac{deta_-}{d\lambda}
ight|_{\lambda=0}\int_0^1 s^{-lpha_-}(-\log s)dh\left(x^*(0)s
ight)-k\leq 0.$$

But in this case, we have  $\alpha_{-} = -1$ ,  $d\beta_{-}/d\lambda|_{\lambda=0} = -2/3$ . Thus

$$\phi(c,k) = \frac{2}{3} \int_0^1 s(-\log s) dh \left( x^*(0)s \right) - k.$$

Thanks to Corollary 5.2, the optimal exercise boundary  $x^*(0)$  for the classical optimization problem  $(\lambda = 0)$  is uniquely determined by equation

$$\int_0^1 h(xs) \, ds = c + k.$$

From these consideration, it is not difficult to compute the function g, at least numerically. The result with M taken as 20, which is reported in Figure 3, shows that the function g is not non-decreasing.

**Summary.** This paper considers a class of optimal stopping problems with multiple entries and forced exits. Explicit solutions are obtained for general payoff functions under mild assumptions. It is discovered that the presence of these forced exits will always raise the optimal exercise boundary and reduce the value function if and only if the entry cost k and the running cost c satisfy  $k \ge g(c)$  for some function g. It is interesting that the function g is not non-decreasing in general. Sufficient conditions for g to be non-decreasing is also given.

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## Appendix A. Collection of Proofs

**Proof of Lemma 3.2.** Suppose that  $\liminf_{x\to\infty} x^{-\beta_+} H_{\lambda}(x) > 0$ . Then it is not difficult to see that there exists a constant  $\varepsilon > 0$  such that

$$\varepsilon \le x^{\beta_--\beta_+} \int_0^x s^{-\beta_--1} h(s) \, ds,$$

for x large enough. Since h is non-decreasing, we have

$$\varepsilon \leq x^{\beta_--\beta_+} \int_0^x s^{-\beta_--1} h(x) \, ds = h(x) x^{-\beta_+}/(-\beta_-).$$

But this implies that

$$\int_x^\infty s^{-\beta_+ - 1} h(s) \, ds \ge -\beta_- \varepsilon \int_x^\infty s^{-\beta_+ - 1} s^{\beta_+} \, ds = \infty,$$

or  $H_{\lambda}(x) = \infty$ , for x large enough. This is a contradiction.

We now show that there exists positive constants  $c_1, c_2$  such that

$$H_{\lambda}(x) \le c_1 x^{\beta_+} + c_2 \tag{A.1}$$

for all x. Indeed, simple computation yields

$$xH'_{\lambda}(x) = \frac{2}{\sigma^{2}(\beta_{+} - \beta_{-})} \left[ \beta_{-} \int_{0}^{1} s^{-\beta_{-} - 1} h(xs) ds + \beta_{+} \int_{1}^{\infty} s^{-\beta_{+} - 1} h(xs) ds \right].$$
(A.2)

In particular, we have  $xH'_{\lambda}(x) \leq \beta_+H_{\lambda}(x)$ . This easily implies that

$$\left[x^{-\beta_+}H_\lambda(x)\right]' \le 0.$$

$$xH'_{\lambda}(x) \le c_1 x^{\beta_+} + c_2. \tag{A.3}$$

It is not difficult to check

$$e^{-(r+\lambda)t}X_t^{\beta_+} = x^{\beta_+}\exp\{\sigma\beta_+W_t - \sigma^2\beta_+^2t/2\}.$$

This and (A.1) clearly imply

$$\lim_{t \to \infty} e^{-(r+\lambda)t} H_{\lambda}(X_t) = 0.$$
(A.4)

As for the second part of the lemma, we observe that Proposition 3.1, the strong Markov property of X and equation (A.4) yield

$$\mathbb{E}^{x}\left[e^{-(r+\lambda)\tau_{n}}H_{\lambda}(X_{\tau_{n}})\right] = \mathbb{E}^{x}\left[\int_{\tau_{n}}^{\infty}e^{-(r+\lambda)t}h(X_{t})\,dt\right].$$

Applying the Dominated Convergence Theorem, we complete the proof.

**Proof of Proposition 4.1.** We first show that equation (4.19) admits a unique strictly positive solution. Let

$$f(x) \doteq \int_0^1 s^{-\beta_- - 1} h(xs) \, ds.$$

Clearly f is non-decreasing. It is not difficult to see that f is also continuous with  $\lim_{x\to 0} f(x) = 0$  from Dominated Convergence Theorem. Furthermore, it follows from Monotone Convergence Theorem that

$$f(\infty) \doteq \lim_{x \to \infty} f(x) = h(\infty) \int_0^1 s^{-\beta_- - 1} ds = -h(\infty)/\beta_-.$$

However, observing that

$$\beta_{+}\beta_{-} = -2(r+\lambda)/\sigma^{2}, \qquad (A.5)$$

we have

$$f(\infty) = \frac{\sigma^2 \beta_+ h(\infty)}{2(r+\lambda)} > \frac{\sigma^2 \beta_+}{2} \left( \frac{c}{r+\lambda} + k \right),$$

which is exactly the right-hand-side of (4.19). The existence of a strictly positive solution to (4.19) is now clear. As for the uniqueness, suppose  $x_1 < x_2$  are both solutions to equation (4.19). Since  $h(x_1s) \leq h(x_2s)$  for every s, we must have  $h(x_1s) = h(x_2s)$  for almost every  $s \in [0, 1]$ . But the continuity of h further implies that  $h(x_1s) = h(x_2s)$  for every  $s \in [0, 1]$ . In particular, we have

$$h(x_2) = h\left(x_2 \cdot \frac{x_1}{x_2}\right) = \dots = h\left(x_2 \cdot \left(\frac{x_1}{x_2}\right)^n\right)$$

for every n. The continuity of h then implies  $h(x_2) = h(0) = 0$ , and whence  $h(x_2s) = 0$  for all  $s \in [0, 1]$ . This is a contradiction since the right-hand-side of (4.19) is strictly positive.

We now show that the three constants  $(A_+, A_-, B_+)$  are all strictly positive. Since h is non-decreasing and non-constant, we have

$$\beta_{-} \int_{0}^{1} s^{-\beta_{-}-1} h(xs) \, ds + \beta_{+} \int_{1}^{\infty} s^{-\beta_{+}-1} h(xs) \, ds$$

$$> h(x) \left[ \beta_{-} \int_{0}^{1} s^{-\beta_{-}-1} \, ds + \beta_{+} \int_{1}^{\infty} s^{-\beta_{+}-1} ds \right]$$

$$= 0.$$

This and equation (A.2) give  $H'_{\lambda}(x) > 0$  for every x > 0, which implies that  $B_+$  is strictly positive. In order to show  $A_+$  is strictly positive, it suffices to show that

$$x^{*}H_{0}'(x^{*}) - \alpha_{-}H_{0}(x^{*}) + \alpha_{-}[c + (r + \lambda)k]/r > 0.$$

It follows from (A.5) and (A.2) (with  $\lambda = 0$ ) and some computation that the above inequality is equivalent to

$$\alpha_{+} \int_{1}^{\infty} s^{-\alpha_{+}-1} h(x^{*}s) \, ds - [c + (r+\lambda)k] > 0.$$

However, we have

$$\alpha_{+} \int_{1}^{\infty} s^{-\alpha_{+}-1} h(x^{*}s) \, ds \ge \alpha_{+} h(x^{*}) \int_{1}^{\infty} s^{-\alpha_{+}-1} \, ds = h(x^{*})$$

It follows trivially from the monotonicity of h, (4.19) and (A.5) that

$$h(x^*) > c + (r + \lambda)k. \tag{A.6}$$

Therefore,  $A_+$  is strictly positive. Similarly, in order to show  $A_- > 0$ , we only need to show

$$\alpha_{-} \int_{0}^{1} s^{-\alpha_{-}-1} h(x^{*}s) \, ds + [c + (r+\lambda)k] > 0.$$

But integration by parts yields

$$\begin{aligned} \alpha_{-} \int_{0}^{1} s^{-\alpha_{-}-1} h(x^{*}s) \, ds &= -h(x^{*}) + \int_{0}^{1} s^{-\alpha_{-}} dh(x^{*}s) \\ &> -h(x^{*}) + \int_{0}^{1} s^{-\beta_{-}} dh(x^{*}s) \\ &= \beta_{-} \int_{0}^{1} s^{-\beta_{-}-1} h(x^{*}s) \, ds. \end{aligned}$$

The desired inequality now follows from (4.19) and (A.5).

It is clear that  $(v, \bar{v}, x^*)$  is the unique solution to the variational inequality, and  $v(x) - \bar{v}(x) = -k$  for all  $x \ge x^*$ . We will show by contradiction that  $v(x) - \bar{v}(x) > -k$  for all  $x \in (0, x^*)$ . Suppose  $v(x) - \bar{v}(x) \le -k$  for some  $x \in (0, x^*)$ . Since  $\lim_{x\to 0} [v(x) - \bar{v}(x)] = c/(r + \lambda) > -k$ , one can always find a  $\bar{x} \in (0, x^*)$  such that

$$v(\bar{x}) - \bar{v}(\bar{x}) \le -k, \quad v'(\bar{x}) - \bar{v}'(\bar{x}) = 0.$$

But this amounts to

$$\begin{aligned} -B_+ \bar{x}^{\beta_+} + H_\lambda(\bar{x}) - c/(r+\lambda) &\geq k \\ -\beta_+ B_+ \bar{x}^{\beta_+} + \bar{x} H'_\lambda(\bar{x}) &= 0. \end{aligned}$$

Multiplying the first inequality by  $\beta_+$  and then subtracting the second equality, we arrive at

$$\beta_+ H_\lambda(\bar{x}) - \bar{x} H'_\lambda(\bar{x}) \ge \beta_+ [c/(r+\lambda) + k].$$

However,

$$\beta_{+}H_{\lambda}(\bar{x}) - \bar{x}H_{\lambda}'(\bar{x}) = \frac{2}{\sigma^{2}}\int_{0}^{1} s^{-\beta_{-}-1}h(\bar{x}s) \, ds.$$

Now by the definition (4.19) of  $x^*$  and that h is non-decreasing, we have  $\bar{x} \ge x^*$ . This is a contradiction.

It remains to show that v is strictly increasing (whence strictly positive). This is trivial on interval  $(0, x^*)$ . As for  $x \ge x^*$ , observe that (4.13) and simple computation yields

$$\begin{aligned} xv'(x) - \alpha_{-}v(x) &= xH'_{0}(x) - \alpha_{-}H_{0}(x) + \alpha_{-}[c + (r + \lambda)k]/r \\ &= \frac{2}{\sigma^{2}}\int_{1}^{\infty} s^{-\alpha_{+}-1}h(xs)ds + \alpha_{-}[c + (r + \lambda)k]/r, \end{aligned}$$

which is clearly non-decreasing. Again, we will show by contradiction that v is non-decreasing on interval  $(x^*, \infty)$ . Suppose this is not the case. We can always find  $x_1 > x_2 > x^*$  such that  $v(x_1) < v(x_2)$ . Since  $v'(x^*) > 0$ , the maximum of v over interval  $[x^*, x_1]$  must be obtained at some interior point  $y_2 \in (x^*, x_1)$ . We have  $y_2 < x_1$ ,  $v(x_1) < v(y_2)$ , and  $v'(y_2) = 0$ . This implies the existence of  $y_1 \in (y_2, \infty)$  such that  $v(y_1) < v(y_2)$  and  $v'(y_1) \leq 0$ . We have

$$y_1v'(y_1) - \alpha_-v(y_1) \le -\alpha_-v(y_1) < -\alpha_-v(y_2) = y_2v'(y_2) - \alpha_-y_2.$$

This contradicts the fact that  $xv'(x) - \alpha_{-}v(x)$  is a non-decreasing function. We complete the proof.

**Proof of Lemma 6.3.** In the proof, we will denote by  $A_{\pm}(\lambda)$  the coefficients  $A_{\pm}$  obtained in equation (4.20), in order to distinguish among different Poisson arrival rates  $\lambda$ . Naturally, we just set  $A_{+}(0)$  as the coefficient  $A_{+}$  in Corollary 5.2. We will consider the following three cases separately.

Case 1:  $x \leq x^*(0)$ . In this case,  $V(x;\lambda) = A_+(\lambda)x^{\alpha_+}$  and  $V(x;0) = A_+(0)x^{\alpha_+}$ , thus we only need to show  $A_+(\lambda) \leq A_+(0)$ . However, thanks to equation (4.18) and that  $A_-(\lambda) \geq 0$  (Proposition 4.1), we have

$$A_{+}(\lambda) \leq [x^{*}(\lambda)]^{-\alpha_{+}+1}H'_{0}(x^{*}(\lambda)).$$

Therefore, it suffices to show that the function

$$f(x) \doteq x^{-\alpha_++1} H_0'(x)$$

is a non-increasing function, or  $f'(x) \leq 0$ . However, direct computation yields that

$$f'(x) = \frac{2}{\sigma^2} \left[ -\alpha_- x^{\alpha_- - \alpha_+ - 1} \int_0^x s^{-\alpha_- - 1} h(s) ds - x^{-\alpha_+ - 1} h(x) \right],$$

which implies, thanks to the monotonicity of h, that

$$f'(x) \le \frac{2}{\sigma^2} \left[ -\alpha_- x^{\alpha_- - \alpha_+ - 1} h(x) \int_0^x s^{-\alpha_- - 1} ds - x^{-\alpha_+ - 1} h(x) \right] = 0.$$

Case 2:  $x \ge x^*(\lambda)$ . In this case,

$$V(x;\lambda) - V(x;0) = A_{-}(\lambda)x^{-\alpha_{-}} - \lambda k/r.$$

Since  $\alpha_{-} < 0$  and  $A_{-}(\lambda) > 0$ , it suffices to show that

$$A_{-}(\lambda)[x^{*}(\lambda)]^{-\alpha_{-}} \leq \lambda k/r,$$

or equivalently,

$$x^*(\lambda)H_0'(x^*(\lambda)) - \alpha_+H_0(x^*(\lambda)) + \alpha_+[c + (r+\lambda)k]/r \le (\alpha_+ - \alpha_-)\lambda k/r.$$

Thanks to equation (A.2), all we need to show is that

$$-\frac{2}{\sigma^2} \int_0^1 s^{-\alpha_- -1} h(x^*(\lambda)s) \, ds + \alpha_+ [c+rk]/r \le -\alpha_- \lambda k/r. \tag{A.7}$$

However, since h is non-decreasing, we have

LHS 
$$\leq -\frac{2}{\sigma^2} \int_0^1 s^{-\alpha_- - 1} h(x^*(0)s) \, ds + \alpha_+ [c + rk]/r = 0,$$

where the equality is from Corollary 5.2. Inequality (A.7) is now immediate.

Case 3:  $x^*(0) < x < x^*(\lambda)$ . Without loss of generality, assume  $x^*(0) < x^*(\lambda)$ . We will show by contradiction. Suppose  $V(x; \lambda) > V(x; 0)$  for some  $x^*(0) < x < x^*(\lambda)$ . Thanks to the results from Case 1 and Case 2, there exists an  $\bar{x} \in (x^*(0), x^*(\lambda))$  such that

$$V(\bar{x};\lambda) - V(\bar{x};0) > 0$$

and

$$V'(\bar{x};\lambda) - V'(\bar{x};0) = 0;$$

here V' is the derivative with respect to x. Since for  $x^*(0) < x < x^*(\lambda)$ ,  $V(x;\lambda) = A_+(\lambda)x^{\alpha_+}$  and  $V(x;0) = H_0(x) - [c+rk]/r$ , we have

$$A_+(\lambda)\bar{x}^{\alpha_+} > H_0(\bar{x}) - [c+rk]/r$$

and

$$\alpha_+ A_+(\lambda) x^{\alpha_+} = \bar{x} H_0'(\bar{x}).$$

It follows that

$$\alpha_{+}H_{0}(\bar{x}) - \alpha_{+}[c+rk]/r < \bar{x}H_{0}'(\bar{x}),$$

or

$$\alpha_{+}H_{0}(\bar{x}) - \bar{x}H_{0}'(\bar{x}) = \frac{2}{\sigma^{2}} \int_{0}^{1} s^{-\alpha_{-}-1}h(\bar{x}s) \, ds < \alpha_{+}[c+rk]/r$$

Thanks to Corollary 5.2 and that h is non-decreasing, we have  $\bar{x} < x^*(0)$ , which is a contradiction.

**Proof of Theorem 6.2.** Thanks to (4.19) and (A.5),  $x^*(\lambda)$  is the unique solution to the equation

$$-\beta_{-}\int_{0}^{1}s^{-\beta_{-}-1}h(xs)\,ds=c+(r+\lambda)k.$$

Clearly,  $x^*(\lambda) \ge x^*(0)$  if and only if

$$-\beta_{-} \int_{0}^{1} s^{-\beta_{-}-1} h\left(x^{*}(0)s\right) \, ds - \lambda k \leq c + rk.$$

Regarding the left-hand-side of the above inequality as a function of  $\lambda$  and denote it by  $f(\lambda)$ , then  $x^*(\lambda) \ge x^*(0)$  if and only if

$$f(\lambda) \le c + rk.$$

We claim that f is a concave function. Indeed, integrating by parts, we arrive at

$$f(\lambda) = h(x^*(0)) - \int_0^1 s^{-\beta_-} dh(x^*(0)s) - \lambda k.$$

However, it is not difficult to check that for any  $s \in [0,1]$ ,  $s^{-\beta_-}$  is a convex function with respect to  $\lambda$ . Therefore,  $f(\lambda)$  is a concave function. Since f(0) = c + rk by Corollary 5.2 and clearly  $\lim_{\lambda \to \infty} f(\lambda) = -\infty$ , there exists a  $\overline{\lambda} \geq 0$  such that  $f(\lambda) \leq c + rk$  if and only if  $\lambda \geq \overline{\lambda}$ . Or equivalently,  $x^*(\lambda) \geq x^*(0)$  if and only if  $\lambda \geq \overline{\lambda}$ . Thanks to Lemma 6.3, we have proved the first part of the theorem.

The preceding proof also implies that  $\overline{\lambda} = 0$  if and only if  $f'(0+) \leq 0$ . However, it follows readily from Dominated Convergence Theorem and the concavity of f that

$$f'(0) = -\left.\frac{d\beta_{-}}{d\lambda}\right|_{\lambda=0} \int_{0}^{1} s^{-\alpha_{-}}(-\log s) dh\left(x^{*}(0)s\right) - k.$$
(A.8)

Regard f'(0) as a function of c and k, say  $\varphi(c, k)$ . It is rather straightforward to see that  $\varphi$  is a continuous function on region  $\{(c, k) : c \ge 0, k \ge 0\}$ , and  $\varphi(c, 0) \ge 0$  with equality if and only if c = 0. We need the following result.

**Lemma A.1** The function  $\varphi(c,k)$  is strictly decreasing with respect to k. It is non-decreasing with respect to c if we further assume  $\bar{h}(x) \doteq h(e^x)$  is convex. Moreover,  $\varphi(c,k) \leq 0$  if  $k \geq 2c/(\sigma\alpha_-)^2$ .

We will assume that Lemma A.1 holds for now, and leave its proof to the end. It follows easily that there exists a function  $g: \mathbb{R}^+ \to \mathbb{R}^+$  such that  $\varphi(c,k) \leq 0$  if and only if  $k \geq g(c)$  with g(0) = 0. Clearly  $g(c) \leq 2c/(\sigma\alpha_-)^2$ . We now prove that g is continuous. Let  $c \in \mathbb{R}^+$  and a sequence of  $\{c_n\}$  such that  $c_n \to c$ . It suffices to show that

$$\limsup_{n} g(c_n) \le g(c) \le \liminf_{n} g(c_n)$$

Let K be an arbitrary non-negative number such that  $K < \limsup_n g(c_n)$ . Then there exist a subsequence, still indexed by n, such that  $K < g(c_n)$ , which implies that  $\varphi(c_n, K) > 0$ , since  $\varphi$  is strictly decreasing in k (Lemma A.1). Therefore, by the continuity of  $\varphi$ , we have

$$\varphi(c, K) = \lim_{n} \varphi(c_n, K) \ge 0.$$

It follows that  $K \leq g(c)$ . Since K is arbitrary, we arrive at  $\limsup_n g(c_n) \leq g(c)$ . The proof of the inequality  $\liminf_n g(c_n) \geq g(c)$  is very similar and thus omitted.

If we further assume that  $\bar{h}(x) = h(e^x)$  is convex, then Lemma A.1 asserts that  $\varphi$  is non-decreasing with respect to c. It follows readily that g is non-decreasing.

**Proof of Lemma A.1.** Equation (A.8) and integration by parts yield

$$\varphi(c,k) = -\left.\frac{d\beta_-}{d\lambda}\right|_{\lambda=0} \int_0^1 s^{-\alpha_- - 1} [1 - \alpha_- \log s] h(x^*(0)s) \, ds - k.$$

Let

$$\varepsilon \doteq - \left. \frac{d\beta_-}{d\lambda} \right|_{\lambda=0} = \frac{1}{\sigma^2} \left[ \left( \frac{1}{2} - \frac{b}{\sigma^2} \right)^2 + \frac{2r}{\sigma^2} \right]^{-1/2},$$

which is strictly positive.

Thanks to Corollary 5.2 and equation (A.5), we have

$$\varphi(c,k) = \varepsilon \left[ \frac{c+rk}{-\alpha_-} - \alpha_- \int_0^1 s^{-\alpha_- - 1} \log s \cdot h(x^*(0)s) \, ds \right] - k.$$

Some algebra yields that

$$\varphi(c,k) = \varepsilon \left[ \frac{c}{-\alpha_-} + \frac{\sigma^2 \alpha_- k}{2} - \alpha_- \int_0^1 s^{-\alpha_- - 1} \log s \cdot h(x^*(0)s) \, ds \right].$$

Since  $\alpha_{-} < 0$ , and  $x^{*}(0)$  increases as k increases, the function  $\varphi$  is clearly strictly decreasing with respect to k. Moreover, it is obvious that

$$arphi(c,k) \leq arepsilon \left[rac{c}{-lpha_{-}} + rac{\sigma^2 lpha_{-} k}{2}
ight].$$

Thus,  $\phi(c,k) \leq 0$  whenever  $k \geq 2c/(\sigma \alpha_{-})^{2}$ .

Finally, assume that  $\bar{h}(x) \doteq h(e^x)$  is convex. Then  $\bar{h}$  is differentiable almost everywhere. Let  $D^+\bar{h}$  denote the right-derivative of  $\bar{h}$ . Then  $D^+\bar{h}$ 

is a right continuous, non-decreasing function. Furthermore, by a change of variable in equation (A.8), we have

$$\begin{split} \phi(c,k) &= \varepsilon \int_{-\infty}^{0} e^{-\alpha_{-}t}(-t)dh\left(x^{*}(0)e^{t}\right) - k \\ &= \varepsilon \int_{-\infty}^{0} e^{-\alpha_{-}t}(-t)d\bar{h}\left(t + \log x^{*}(0)\right) - k \\ &= \varepsilon \int_{-\infty}^{0} e^{-\alpha_{-}t}(-t)D^{+}\bar{h}\left(t + \log x^{*}(0)\right)dt - k \end{split}$$

But as c increases, so does  $x^*(0)$ . Since  $D^+\bar{h}$  in non-decreasing for every t, we complete the proof.



Figure 1: Effect of forced exits on optimal exercise boundaries



Figure 2: Effect of forced exits on value functions



Figure 3: The function g is not always non-decreasing