Some Control Problems with Random Intervention Times

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Abstract

We consider the problem of optimally tracking a Brownian motion by a sequence of impulse controls, in such a way to minimize the total expected cost that consists of a quadratic deviation cost and a proportional control cost. The main feature of our model is that the control can only be exerted at the arrival times of an exogenous uncontrolled Poisson process (signal). In other words, the set of possible intervention times are discrete, random and determined by the signal process (not by the decision maker). We discuss both the discounted problem and the ergodic problem, where explicit solutions can be found. We also derive the asymptotic behavior of the optimal control policies and the value functions as the intensity of the Poisson process goes to infinity, or roughly speaking, as the set of admissible controls goes from the discrete-time impulse control to the continuous-time bounded variation control.

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1 Introduction

In this paper we consider a control problem in which the state evolution is modeled as follows:

$$X_t = x + W_t + \xi_t, \qquad X_0 = x; \tag{1.1}$$

here $x \in \mathbb{R}$, $W = \{W_t; t \geq 0\}$ is a standard Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathsf{P})$ with filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions: right-continuity and completion by P-negligible sets. The class of *admissible controls*, denoted by $\mathcal{B}$, consists of those left-continuous processes $\xi = \{\xi_t; t \geq 0\}$ that have the following representation:

$$\xi_t = \int_{[0,t)} \theta_s dN_s; \tag{1.2}$$

here $N = \{N_t; t \geq 0\}$ is an $\{\mathcal{F}_t\}$-adapted Poisson process with intensity λ, and $\theta = \{\theta_t; t \geq 0\}$ is assumed to be $\{\mathcal{F}_t\}$-predictable. In other words, controls can only be exerted at the arrival times of the Poisson process N. We adopt the following assumption throughout the paper.

Assumption: The Brownian motion W and the Poisson process N are independent.

The objective is to minimize the expected total discounted cost (discount problem)

$$\mathsf{E} \int_0^\infty e^{-\alpha t} \left(X_t^2 \, dt + c \, d\check{\xi}_t \right), \tag{1.3}$$

or the average cost per unit time (ergodic problem)

$$\liminf_{T \to \infty} \frac{1}{T} \mathsf{E} \int_0^T \left(X_t^2 \, dt + c \, d\check{\xi}_t \right), \tag{1.4}$$

over all the admissible control strategies $\xi \in \mathcal{B}$. Here α, c are both positive numbers, and $\check{\xi} = \{\check{\xi}_t; t \geq 0\}$ is the *total variation process* of ξ, that is

$$\check{\xi}_t \triangleq \int_0^t |\theta_s| \, dN_s \quad \text{or} \quad d\check{\xi}_t = |d\xi_t| \, . \tag{1.5}$$

One main feature of this model is that the control policy is essentially *discrete*, in a sense that the control can only be exerted at the times when the Poission process N has a jump. In other words, controls can only be adjusted at the discrete *random* times $0 < T_1 < T_2 < T_3 < \cdots$ where $\{T_1, T_2 - T_1, T_3 - T_2, \cdots\}$ are independent identically distributed (iid) exponential random variables with rate λ. The other main feature is that the possible intervention times are completely determined by the exogenous, *uncontrolled* Poisson process N. In other words, the decision maker cannot intervene the system freely — he has to rely on the Poisson process to gives him a certain signal (in this case, the jumps), in order to impose controls.

Control problems of similar type, but with control policies that can be adjusted instantaneously and continuously, have been extensively studied in the last three decades and have found applications in many areas like engineering, economics, finance and biology, etc. A very partial list of references includes [1], [2], [3], [6], [8], [11], [12], [14], [21]. One major attractive aspect of this formulation lies in the possibility of obtaining explicit solutions, especially when the time horizon is infinite. However, the optimal strategies in such occasions are often *singular* with respect to the Lebesgue measure, which makes them very hard to implement in practice. An alternative type of control problems allow the control policies to be adjusted only at the times which are multiples of some fixed positive number; e.g., [9], [13], [18]. While this discrete-time formulation seems to be more realistic, it is usually very difficult (sometimes impossible) to obtain explicit solutions even when the time horizon is infinite. This drawback makes the subsequent analysis much more intractable. Another type of control is the so-called impulse control. The decision makers are allowed to choose a sequence of stopping times (intervention times) $\{\tau_1, \tau_2, \cdots\}$, and a sequence of impulses controls $\{\zeta_1, \zeta_2, \cdots\}$ to be imposed upon the system at $\{\tau_1, \tau_2, \cdots\}$ respectively; e.g. [4], [10], [15], [16]. Explicit solutions are possible for simple cases with infinite time horizon.

Our formulation will allow us to obtain explicit solutions for some simple models with infinite time horizon, while the underlying control strategies are kept discrete. The difference from the usual impulse control problems is that the intervention times are no longer a total freedom to the decision maker – they are essentially determined by the exogenous signal process (the Poisson process N). Such kind of constraints on the intervention times, to the best knowledge of the author, were first used in [19] as a simplified model for liquidity effects.

This paper is orginised as follows. In section 2 we solve the discounted problem (1.3). We show that there exist certain thresholds $\pm x^*$ depending on the parameters, such that it is optimal to control when the state process exceeds x^* or falls below $-x^*$ at Poisson jumps, and then exert exactly the amount needed to push the state process back to the nearest threshold. Similar strategy (with possibly different thresholds) is also optimal for the control problem (1.4) minimizing average cost per unit time, as we shall see in Section 3. The connection between the discounted problem and the ergodic problem is discussed in Section 4. Asymptotic analysis is carried out in Section 5 as λ (the intensity of the underlying Poisson process N) goes to infinity, in order to compare with the usual singular control model where the control policies can be adjusted continuously. We found the value functions and the optimal threshholds converge with rates $\frac{1}{\lambda}$ and $\sqrt{\frac{1}{\lambda}}$ respectively. For completeness, a brief account for the singular control problems is also included in the section.

Remark 1. *The optimal strategies in both problems invlove jumping to the nearest boundary of a certain interval $[-a, a]$ at the times of Poisson jumps. To ease exposition, we introduce the function*

$$L(x; a) \triangleq (x + a)^- - (x - a)^+ = \left\{ \begin{array}{ccc} a - x & ; & x > a \\ 0 & ; & -a \leq x \leq a \\ -x - a & ; & x < -a \end{array} \right\}. \tag{1.6}$$

Here $x^\pm$ denote the positive and negative parts of $x \in \mathbb{R}$, respectively. It is easy to see that $L(x; a)$ is the exact amount of control needed to push the state process to the nearest boundary of the interval $[-a, a]$, provided the current state is x.

2 The discounted problem

For every control process $\xi \in \mathcal{B}$, we associate with the expected total discounted cost

$$J(x;\xi) \stackrel{\triangle}{=} \mathsf{E}\int_0^\infty e^{-\alpha t}\left(X_t^2 dt + c\, d\check{\xi}_t\right);$$

here α, c are arbitrary positive constants. The objective is to minimize $J(x;\xi)$ over all $\xi \in \mathcal{B}$.

Let us proceed heuristically for a while. Let $v(x)$ be the value function. It follows from the Dynamic Programming Principle that the process

$$Y_t \stackrel{\triangle}{=} \int_0^t e^{-\alpha s}\left(X_s^2\, ds + c\, d\check{\xi}_s\right) + e^{-\alpha t} v(X_t)$$

is indeed a submartingale. However, assuming that the value function $v(x)$ is twice continuously differentiable, the generalized Itô formula yields

$$dY_t = dM_t + e^{-\alpha t}\left[X_t^2 - \alpha v(X_t) + \frac{1}{2}v''(X_t) + \lambda\left(v(X_t + \triangle\xi_t) + c\,|\triangle\xi_t| - v(X_t)\right)\right] dt,$$

where $M = \{M_t;\ t \geq 0\}$ is some local martingale (see any of [5], [17], [20] for more background on stochastic calculus for processes with jumps). An intutive explaination for the last term is that the Poisson process has probability $\lambda\, dt$ to have a jump in a small time interval of length dt, or the process will jump from X_t to $X_t + \triangle\xi_t$ with probability $\lambda\, dt$. One would naturally expect the value function $v(x)$ to satisfy the equation

$$\min_{\theta\in\mathbb{R}}\left[x^2 - \alpha v(x) + \frac{1}{2}v''(x) + \lambda\big(v(x+\theta) + c|\theta| - v(x)\big)\right] = 0.$$

However, it is very easy to see that the value function $v(x)$ is even-symmetric and convex, therefore

$$\min_{\theta\in\mathbb{R}}\big(v(x+\theta) + c|\theta|\big)$$

is achieved at $\theta = 0$ if $x \in [-x^*, x^*]$, and achieved at $\theta = x^* - x$ if $x > x^*$ or at $\theta = -x^* - x$ if $x < -x^*$. Here x^* is determined by the equation $v'(x^*) = c$ (hence $v'(-x^*) = -c$). Finally, observe that

$$0 \leq v(x) \leq J(x;0) = \mathsf{E}\int_0^\infty e^{-\alpha t}(x + W_t)^2\, dt = \frac{1}{\alpha}x^2 + \frac{1}{\alpha^2}.$$

We obtain the following variational inequality from the above heuristic arguments.

Variational Inequality: Find a nonnegative, twice continuously differentiable, even-symmetric convex function $v(x)$, and a constant $x^* > 0$ such that $v(x) = O(x^2)$ as $x \to \pm\infty$, and

$$v'(0) = 0; \tag{2.1}$$

$$v'(x^*) = c; \tag{2.2}$$

$$\frac{1}{2}v''(x) - \alpha v(x) + x^2 = 0; \qquad 0 \leq x < x^* \tag{2.3}$$

$$\frac{1}{2}v''(x) - (\alpha+\lambda)v(x) + x^2 + \lambda\left(v(x^*) + c(x - x^*)\right) = 0; \qquad x \geq x^* \tag{2.4}$$

By even-symmetry, $v(x)$ can be extended to $x < 0$.

This variational inequality admits a unique solution that can be calculated explicitly. It turns out that the solution is indeed the value function, and the optimal strategy can be described as follows.

Optimal Strategy: Do not act as long as the state process stays in the region $[-x^*, x^*]$. However, exert exact amount of control to push the state process back to the nearest boundary ($\pm x^*$ respectively), if the state process is outside the region $[-x^*, x^*]$ at the Poisson arrival times.

We have the following theorem.

Theorem 1. *Let* $(v(x), x^*)$ *be the unique solution to the variational inequality* (2.1) – (2.4). *We have*

$$v(x) = \inf_{\xi \in \mathcal{B}} J(x; \xi) = \inf_{\xi \in \mathcal{B}} \mathsf{E} \int_0^\infty e^{-\alpha t} \left(X_t^2 \, dt + c \, d\check{\xi}_t \right).$$

Moreover, the optimal strategy $\{\xi_t^* = \int_{[0,t)} \theta_s^* \, dN_s; \ t \geq 0\}$ *can be determined inductively by*

$$\theta_t^* = L(x + W_t + \xi_t^*; \ x^*) = L(x + W_t + \int_{[0,t)} \theta_s^* \, dN_s; \ x^*) \tag{2.5}$$

in the notation of (1.6).

The rest of the section is devoted to solving the variational inequality (2.1) – (2.4) and proving Theorem 1. We have to show that the solution $(v(x), x^*)$ is unique, and the conjectured strategy is indeed optimal.

2.1 Solution to the variational inequality

To solve the variational inequality (2.1) – (2.4), we first observe that equation (2.3) implies that

$$v(x) = A_1 e^{\sqrt{2\alpha} x} + A_2 e^{-\sqrt{2\alpha} x} + \frac{1}{\alpha} x^2 + \frac{1}{\alpha^2}; \qquad 0 \leq x \leq x^*$$

for some constant A_1, A_2. However, it follows from equation (2.1) that $A_1 = A_2$, which in turn implies that, with $A \triangleq 2A_1$,

$$v(x) = A \cosh(\sqrt{2\alpha} x) + \frac{1}{\alpha} x^2 + \frac{1}{\alpha^2}; \qquad 0 \leq x \leq x^*. \tag{2.6}$$

For $x > x^*$, it is not very difficult to verify that equation (2.4) implies

$$v(x) = B e^{-\sqrt{2(\alpha+\lambda)} x} + C e^{\sqrt{2(\alpha+\lambda)} x} + \frac{1}{\alpha+\lambda} x^2 + \frac{c\lambda}{\alpha+\lambda} x + \frac{1}{(\alpha+\lambda)^2} + \frac{\lambda}{\alpha+\lambda} \big(v(x^*) - c x^* \big).$$

However, we must have $C = 0$ since $v(x)$ is non-negative and $v(x) = O(x^2)$ as $x \to \infty$. Therefore,

$$v(x) = B e^{-\sqrt{2(\alpha+\lambda)} x} + \frac{1}{\alpha+\lambda} x^2 + \frac{c\lambda}{\alpha+\lambda} x + \frac{1}{(\alpha+\lambda)^2} + \frac{\lambda}{\alpha+\lambda} \big(v(x^*) - c x^* \big); \qquad x > x^*. \tag{2.7}$$

There are three unknowns (A, B, x^*). However, the continuity of $v(x)$ and $v'(x)$, as well as equation (2.2), gives

$$v(x^*+) = v(x^*-), \qquad v'(x^*+) = v'(x^*-) = c.$$

In other words, we have

$$\begin{aligned}
A\cosh(\sqrt{2\alpha}x^*) &= \frac{\alpha+\lambda}{\alpha} Be^{-\sqrt{2(\alpha+\lambda)}x^*} - \frac{\lambda}{\alpha^2(\alpha+\lambda)} \\
A\sinh(\sqrt{2\alpha}x^*) &= -\frac{2}{\alpha\sqrt{2\alpha}}\left(x^* - \frac{c\alpha}{2}\right) \\
Be^{-\sqrt{2(\alpha+\lambda)}x^*} &= \frac{2}{(\alpha+\lambda)\sqrt{2(\alpha+\lambda)}}\left(x^* - \frac{c\alpha}{2}\right)
\end{aligned}$$

It is very easy to derive that x^* should satisfy the following equation:

$$0 = \left(\frac{2}{\sqrt{2(\alpha+\lambda)}} + \frac{2}{\sqrt{2\alpha}}\coth(\sqrt{2\alpha}x^*)\right)\left(x^* - \frac{c\alpha}{2}\right) - \frac{\lambda}{\alpha(\alpha+\lambda)} := g(x^*) \tag{2.8}$$

and the pair of constants (A, B) are conveniently determined by

$$A = -\frac{1}{\sinh(\sqrt{2\alpha}x^*)}\frac{2}{\alpha\sqrt{2\alpha}}\left(x^* - \frac{c\alpha}{2}\right); \quad B = \frac{2}{(\alpha+\lambda)\sqrt{2(\alpha+\lambda)}}\left(x^* - \frac{c\alpha}{2}\right)e^{\sqrt{2(\alpha+\lambda)}x^*}. \tag{2.9}$$

We have the following proposition.

Proposition 1. *Equation* (2.8) *uniquely determines $x^* > 0$, and the function $v(x)$ determined by equations* (2.6), (2.7), (2.9) *is the unique solution to the variational inequality* (2.1)–(2.4).

Proof: We deduce that the function $g(x)$ is strictly increasing for $x > 0$ since $\coth x$ is strictly decreasing and $x\coth x$ is strictly increasing for $x > 0$. The existence and uniqueness of the positive solution x^* to equation (2.8) follow immediately from

$$\lim_{x\to 0} g(x) = -\infty, \qquad \lim_{x\to\infty} g(x) = +\infty.$$

Indeed, we have $x^* > \frac{c\alpha}{2}$ since

$$g\left(\frac{c\alpha}{2}\right) = -\frac{\lambda}{\alpha(\alpha+\lambda)} < 0.$$

The twice continuous differentiability of function $v(x)$ follows readily from (2.3) and (2.4). It remains to show that $v(x)$ is convex and non-negative. However, since $x^* > \frac{c\alpha}{2}$, it follows from (2.9) that $B > 0, A < 0$. Therefore, function $v(x)$ is convex for $x > x^*$. As for $0 \le x < x^*$, we have

$$\begin{aligned}
v''(x) &= 2\left(\alpha A\cosh(\sqrt{2\alpha}x) + \frac{1}{\alpha}\right) \ge 2\left(\alpha A\cosh(\sqrt{2\alpha}x^*) + \frac{1}{\alpha}\right) \\
&= 2\left(\frac{2}{\sqrt{2(\alpha+\lambda)}}\left(x^* - \frac{c\alpha}{2}\right) + \frac{1}{\alpha+\lambda}\right) \ge 0.
\end{aligned}$$

It follows that $v(x)$ is convex on $x \geq 0$. In particular,

$$v(0) = \frac{1}{2\alpha} v''(0) \geq 0,$$

which implies the non-negativity of $v(x)$, thanks to (2.1), (2.3). We complete the proof. □

Remark 2. *The solution $v(x)$ satisfies the equation*

$$-\alpha v(x) + \frac{1}{2} v''(x) + x^2 + \lambda \min_{\theta \in \mathbb{R}} \big(v(x+\theta) + c|\theta| - v(x) \big) = 0.$$

for all $x \in \mathbb{R}$.

2.2 Proof of Theorem 1

First we show that $J(x;\xi) \geq v(x)$ for all the admissible control processes $\xi \in \mathcal{B}$. It suffices to show this inequality for all $\xi \in \mathcal{B}$ such that

$$\mathsf{E} \int_0^\infty e^{-\alpha t} X_t^2 \, dt < \infty. \tag{2.10}$$

In the following, we will repeatedly utilize the following inequalities, whose proof is straightforward and thus omitted:

$$v(x) \leq \varepsilon_1 x^2 + \varepsilon_2; \qquad |v'(x)| \leq \varepsilon_3 |x| + \varepsilon_4 \tag{2.11}$$

for all $x \in \mathbb{R}$ and some positive constants $\varepsilon_i, i = 1, \cdots, 4$.

Applying the Doléan-Dade-Meyer formula to the process $\{e^{-\alpha t} v(X_t); t \geq 0\}$, we obtain that

$$\begin{aligned} e^{-\alpha T} v(X_T) - v(x) &= \int_0^T e^{-\alpha t} \big[-\alpha v(X_t) + \frac{1}{2} v''(X_t) \big] \, dt + \int_0^T e^{-\alpha t} v'(X_t) \, dW_t \\ &\quad + \sum_{0 \leq t \leq T} e^{-\alpha t} \big[v(X_t + \triangle \xi_t) - v(X_t) \big] \end{aligned}$$

almost surely for every $T \geq 0$. However, since $\triangle \xi_t \neq 0$ only if there is a Poisson jump at time t, we have

$$v(X_t + \triangle \xi_t) - v(x) + c|\triangle \xi_t| \geq g(X_t) \, dN_t$$

where

$$g(x) \stackrel{\triangle}{=} \min_{\theta \in \mathbb{R}} \big(v(x+\theta) + c|\theta| - v(x) \big).$$

A bit algebra shows, thanks to Remark 2, that

$$e^{-\alpha T} v(X_T) + \int_0^T e^{-\alpha t} \left(X_t^2 \, dt + c \, d\check{\xi}_t \right) \geq v(x) + M_T + Z_T, \tag{2.12}$$

where $M = \{M_t; t \geq 0\}$ and $Z = \{Z_t; t \geq 0\}$ are local martingale terms defined as

$$M_T \triangleq \int_0^T e^{-\alpha t} v'(X_t)\, dW_t; \qquad Z_T \triangleq \int_0^T e^{-\alpha t} g(X_t) d\tilde{N}_t.$$

Here $\tilde{N} = \{N_t - \lambda t; t \geq 0\}$ is the compensated Poisson process. However, it follows from (2.10) and (2.11) that M is indeed a true martingale. In particular, $\mathsf{E} M_T = 0$. As for the other local martingale term Z, we claim that it is uniformly bounded from above by an integrable random variable. Actually,

$$\int_0^T e^{-\alpha t} g(X_t) d\tilde{N}_t \leq \int_0^\infty e^{-\alpha t} \lambda v(X_t)\, dt \leq \int_0^\infty \lambda e^{-\alpha t} \left(\varepsilon_1 X_t^2 + \varepsilon_2\right)\, dt$$

since $0 \geq g(x) \geq -v(x)$, $dN_t \geq 0$ and (2.11). It follows that Z is in fact a submartingale. In particular, $\mathsf{E} Z_T \geq 0$. Therefore, taking expectation on both sides of (2.12), we have shown that

$$e^{-\alpha T} \mathsf{E} v(X_T) + \mathsf{E} \int_0^T e^{-\alpha t} \left(X_t^2\, dt + c\, d\check{\xi}_t\right) \geq v(x) \tag{2.13}$$

for all $T \geq 0$. Letting $T \to \infty$, we obtain

$$v(x) \leq \liminf_{T\to\infty} e^{-\alpha T} \mathsf{E} v(X_T) + \mathsf{E} \int_0^\infty e^{-\alpha t} \left(X_t^2\, dt + c\, d\check{\xi}_t\right) \leq \liminf_{T\to\infty} e^{-\alpha T} \mathsf{E} v(X_T) + J(x;\xi).$$

However, observe that $\liminf_{T\to\infty} e^{-\alpha T} \mathsf{E} v(X_T) = 0$. Otherwise we have

$$\varepsilon_1 \liminf_{T\to\infty} e^{-\alpha T} \mathsf{E} X_T^2 = \liminf_{T\to\infty} e^{-\alpha T} \mathsf{E}(\varepsilon_1 X_T^2 + \varepsilon_2) \geq \liminf_{T\to\infty} e^{-\alpha T} \mathsf{E} v(X_T) > 0,$$

which violates assumption (2.10), thanks to inequality (2.11). It follows readily that $J(x;\xi) \geq v(x)$ for all $\xi \in \mathcal{B}$.

It remains to show that $v(x) = J(x;\xi^*)$. We only need to show that $J(x;\xi^*) \leq v(x)$. Actually, it is very easy to see that inequality (2.12) is indeed an equality as $\xi = \xi^*$, which in turn implies that the local martingale term $M + Z$ is uniformly bounded from below by $-v(x)$. Therefore, $M + Z$ is a supermartingale. In particular, $\mathsf{E}\big[M_T + Z_T\big] \leq 0$, which implies that

$$\mathsf{E} \int_0^T e^{-\alpha t} \left(X_t^2\, dt + c\, d\check{\xi}_t^*\right) \leq e^{-\alpha T} \mathsf{E} v(X_T) + \mathsf{E} \int_0^T e^{-\alpha t} \left(X_t^2\, dt + c\, d\check{\xi}_t^*\right) \leq v(x).$$

Letting $T \to \infty$, we have $J(x;\xi^*) \leq v(x)$. This completes the proof. □

3 The ergodic problem

Let us consider the following ergodic cost criterion as defined in (1.4):

$$Q(x;\xi) \triangleq \liminf_{T\to\infty} \frac{1}{T} \mathsf{E} \int_0^T \left(X_t^2\, dt + c\, d\check{\xi}_t\right).$$

The objective is to minimize $Q(x;\xi)$ (average cost per unit time) over all admissible controls $\xi \in \mathcal{B}$.

Like before, let us proceed heuristically for a while, and denote the minimum average cost by β (it is very easy to see that this minimum does not depend on the initial state $X_0 = x$). Define

$$v(T,x) \triangleq \inf_{\xi \in \mathcal{B}} \mathsf{E} \int_0^T \left(X_t^2 \, dt + c \, d\check{\xi}_t \right).$$

(see [7] for an excellent heuristic discussion on the ergodic control over general Markov diffusions). At least formally, $v(t,x)$ will satisfy the equation

$$-\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + x^2 + \lambda \min_{\theta \in \mathbb{R}} \big(v(t, x+\theta) + c|\theta| - v(t,x) \big) = 0.$$

Let us use the heuristic $v(t,x) \sim \beta t + V(x)$ for some non-negative function $V(x)$. Then (V, β) will formally satisfy the equation

$$-\beta + \frac{1}{2} V''(x) + x^2 + \lambda \min_{\theta \in \mathbb{R}} \big(V(x+\theta) + c|\theta| - V(x) \big) = 0.$$

However, since $v(t,x)$ is convex and even-symmetric with respect to x, so is $V(x)$. It follows that

$$\min_{\theta \in \mathbb{R}} \big(V(x+\theta) + c|\theta| - V(x) \big)$$

is achieved at $\theta = 0$ if $x \in [-b, b]$, and achieved at $\theta = b - x$ if $x > b$ or at $\theta = -b - x$ if $x < -b$. Here b is determined by the equation $V'(b) = c$. We have the following variational inequality.

Variational Inequality: Find a nonnegative, twice continuously differentiable, even-symmetric convex function $V(x)$ and constants $b, \beta > 0$, such that $V(x) = O(x^2)$ as $x \to \pm\infty$, and

$$V'(0) = 0; \tag{3.1}$$

$$V'(b) = c; \tag{3.2}$$

$$\frac{1}{2} V''(x) + x^2 = \beta; \qquad 0 \le x < b \tag{3.3}$$

$$\frac{1}{2} V''(x) - \lambda V(x) + x^2 + \lambda \left(V(b) + c(x-b) \right) = \beta; \qquad x \ge b \tag{3.4}$$

By even-symmetry, $V(x)$ can be extended to $x < 0$.

This variational inequality can be solved explicitly. One can show that the constants (b, β) are uniquely determined by the variational inequality. One can also show that β is indeed the minimum average cost, while the optimal strategy is the same as the discounted problem but with possibly different barriers.

Optimal Strategy: Do not act as long as the state process stays in the region $[-b, b]$. However, exert exact amount of control to push the state process back to the nearest boundary ($\pm b$ respectively), if the state process is outside the region $[-b, b]$ at the times of Poisson jumps.

Note that $V(x)$ is only uniquely determined by the variational inequality up to a constant (this is also implicitly implied in our heuristic argument); see Proposition 2 for more details.

We have the following result.

Theorem 2. *Let* $(V(x), b; \beta)$ *be a solution to the variational inequality* (3.1) – (3.4)*. We have*

$$\beta = \inf_{\xi \in \mathcal{B}} Q(x; \xi) = \inf_{\xi \in \mathcal{B}} \liminf_{T \to \infty} \frac{1}{T} \mathsf{E} \int_0^T \left(X_t^2 \, dt + c \, d\check{\xi}_t \right).$$

Moreover, the optimal strategy $\{(\xi_*)_t = \int_{[0,t)} (\theta_*)_s \, dN_s;\ t \geq 0\}$ *can be determined inductively by*

$$(\theta_*)_t = L(x + W_t + (\xi_*)_t;\ b) = L(x + W_t + \int_{[0,t)} (\theta_*)_s \, dN_s;\ b) \tag{3.5}$$

in the notation of (1.6)*.*

Below we should first solve the variational inequality (3.1) – (3.4), and then prove Theorem 2.

Remark 3. *It follows from the proof of Theorem 2 that for the optimal control process* ξ_**, the "liminf" in Theorem 2 is indeed a true limit, that is,*

$$\beta = Q(x; \xi_*) = \lim_{T \to \infty} \mathsf{E} \frac{1}{T} \int_0^T \left(X_t^2 + c \, d \left(\check{\xi}_* \right)_t \right) = \inf_{\xi \in \mathcal{B}} Q(x; \xi).$$

3.1 Solution to the variational inequality

Equations (3.3) and (3.1) imply that

$$V(x) = -\frac{1}{6} x^4 + \beta x^2 + A; \qquad 0 \leq x < b \tag{3.6}$$

for some constant A, while equation (3.4) and the condition $V(x) = O(x^2)$ imply that

$$V(x) = B e^{-\sqrt{2\lambda} x} + \frac{1}{\lambda} x^2 + cx + \big(V(b) - cb \big) + \frac{1}{\lambda^2} - \frac{\beta}{\lambda}; \qquad x \geq b \tag{3.7}$$

for some constant B. However, the continuity of $V(x)$ at $x = b$ yields that

$$B e^{-\sqrt{2\lambda} b} + \frac{1}{\lambda} b^2 + \frac{1}{\lambda^2} - \frac{\beta}{\lambda} = 0. \tag{3.8}$$

Furthermore, equation (3.2) is equivalent to $V'(b+) = V'(b-) = c$, or,

$$-\frac{2}{3} b^3 + 2\beta b \;=\; c, \tag{3.9}$$

$$-\sqrt{2\lambda} B e^{-\sqrt{2\lambda} b} + \frac{2}{\lambda} b \;=\; 0. \tag{3.10}$$

Using (3.8) and (3.10) to cancel B, we have

$$b^2 + \frac{1}{\lambda} - \beta + \frac{2}{\sqrt{2\lambda}} b = 0,$$

which, combined with (3.9), gives the equation for b

$$0 = \frac{2}{3} b^2 + \frac{1}{\lambda} - \frac{c}{2b} + \frac{2}{\sqrt{2\lambda}} b := g(b). \tag{3.11}$$

We will show below that this equation uniquely determines $b > 0$, which in turn uniquely determines that

$$\beta = \frac{1}{3} b^2 + \frac{c}{2b}, \tag{3.12}$$

thanks to equation (3.9). Finally, it follows from (3.10) that

$$B = \frac{2}{\lambda \sqrt{2\lambda}} b e^{\sqrt{2\lambda} b}. \tag{3.13}$$

We have the following result.

Proposition 2. *The pair of positive constants* (b, β) *are uniquely determined by equations* (3.11) *and* (3.12)*. Moreover, for any* $A \geq 0$*,* $V(x)$ *given by equations* (3.6)*,* (3.7) *and* (3.13) *is a solution to the variational inequality* (3.1)*–*(3.4)*.*

Proof: The function $g(x)$ defined in (3.11) is strictly increasing with

$$\lim_{x \to 0} g(x) = -\infty, \qquad \lim_{x \to \infty} g(x) = \infty.$$

It follows that $b > 0$ is uniquely determined by equation (3.11). In particular, we have

$$\frac{2}{3} b^2 - \frac{c}{2b} < 0 \quad \Rightarrow \quad c > \frac{4}{3} b^3. \tag{3.14}$$

For any non-negative constant A, it is not difficult to verify that $V(x)$ given by (3.6)–(3.7) is twice continuously differentiable, and convex for $x \geq b$ (note $B > 0$). It remains to show that $V(x)$ is convex on $[0, b]$. However, for $0 \leq x \leq b$,

$$V''(x) = -2x^2 + 2\beta \geq -2b^2 + 2\beta = -2b^2 + 2\left(\frac{1}{3} b^2 + \frac{c}{2b}\right) = 2\left(-\frac{2}{3} b^2 + \frac{c}{2b}\right) \geq 0,$$

thanks to (3.12), (3.14). Furthermore, $V(x) \geq 0$ for all $x \in \mathbb{R}$ since $V(0) = A \geq 0$ and $V'(0) = 0$. □

Remark 4. *The solution* $\big(V(x), b; \beta\big)$ *satisfies the equation*

$$\frac{1}{2} V''(x) + x^2 + \lambda \min_{\theta \in \mathbb{R}} \big(V(x + \theta) + c|\theta| - V(x)\big) = \beta.$$

for all $x \in \mathbb{R}$.

3.2 Proof of Theorem 2

Part of the proof of Theorem 2 is very similar to that of Theorem 1. Fix any $A \geq 0$, which gives a solution to the variational inequality (3.1) – (3.4). First observe that inequalities similar to (2.11) still hold here, that is,

$$V(x) \leq \epsilon_1 x^2 + \epsilon_2; \quad |V'(x)| \leq \epsilon_3 |x| + \epsilon_4 \tag{3.15}$$

for all $x \in \mathbb{R}$ and some positive constants $\epsilon_i, i = 1, \cdots, 4$.

For any admissible control process $\xi \in \mathcal{B}$, we apply the Doléan-Dade-Meyer formula to the process $\{V(X_t); t \geq 0\}$ to obtain

$$V(X_T) - V(x) = \int_0^T \frac{1}{2} V''(X_t)\, dt + \int_0^T V'(X_t)\, dW_t + \sum_{0 \leq t \leq T} \big[V(X_t + \triangle \xi_t) - V(X_t)\big].$$

Define the function

$$g(x) \triangleq \min_{\theta \in \mathbb{R}} \big(V(x+\theta) + c|\theta| - V(x)\big).$$

Similar to the proof of (2.12), we have

$$V(X_T) + \int_0^T \big(X_t^2\, dt + c\, d\check{\xi}_t\big) \geq \beta T + V(x) + M_T + Z_T \tag{3.16}$$

almost surely, thanks to Remark 4. Here M and Z are two local martingale terms defined as

$$M_T \triangleq \int_0^T V'(X_t)\, dW_t, \qquad Z_T \triangleq \int_0^T g(X_t) d\tilde{N}_t,$$

with $\tilde{N}$ standing for the compensated Poisson process. We want to show that

$$\mathsf{E} V(X_T) + \mathsf{E} \int_0^T \big(X_t^2\, dt + c\, d\check{\xi}_t\big) \geq \beta T + V(x). \tag{3.17}$$

To this end, it suffices to show (3.17) for all $\xi \in \mathcal{B}$ with $\mathsf{E} \int_0^T X_t^2\, dt$ being finite. The proof is exactly the same as that of (2.13) — for all such ξ, it follows from (3.15) that M is actually a martingale and that Z is a local martingale uniformly bounded from above by an integrable random variable – hence a submartingale. In particular, $\mathsf{E}\,[M_T + Z_T] \geq 0$. Inequality (3.17) follows readily by taking expectation on both sides of (3.16). Now dividing both sides of (3.17) by T, and then letting $T \to \infty$, we have

$$\liminf_{T \to \infty} \left[\mathsf{E} \frac{1}{T} V(X_T) + \frac{1}{T} \mathsf{E} \int_0^T \big(X_t^2\, dt + c\, d\check{\xi}_t\big) \right] \geq \beta.$$

It is easy to deduce from (3.15) that

$$\liminf_{T \to \infty} \left[\mathsf{E} \frac{\epsilon_1}{T} X_T^2 + \frac{1}{T} \mathsf{E} \int_0^T \big(X_t^2\, dt + c\, d\check{\xi}_t\big) \right] \geq \beta \tag{3.18}$$

must hold. We claim that (3.18) implies that

$$Q(x;\xi) = \liminf_{T\to\infty} \frac{1}{T}\mathsf{E}\int_0^T \left(X_t^2\,dt + c\,d\check{\xi}_t\right) \geq \beta \tag{3.19}$$

for all $\xi \in \mathcal{B}$. The proof of this claim is elementary but a bit technical, hence we will leave it to the end of this subsection and proceed undistracted.

Suppose that (3.19) holds. It remains to show that $Q(x;\xi_*) = \beta$. We only need to show that $Q(x;\xi_*) \leq \beta$. Indeed, inequality (3.16) will become equality if $\xi = \xi_*$. It follows that the local martingale term $M + Z$ now is uniformly bounded from below by $-V(x) - \beta T$, hence a supermartingale. In particular, $\mathsf{E}\,[M_T + Z_T] \leq 0$. Therefore

$$\mathsf{E}\int_0^T \left(X_t^2\,dt + c\,d\left(\check{\xi}_*\right)_t\right) \leq \beta T + V(x) - \mathsf{E}V(X_T) \leq \beta T + V(x).$$

Dividing both sides by T, and then letting $T \to \infty$, we have

$$Q(x;\xi_*) \leq \limsup_{T\to\infty} \frac{1}{T}\mathsf{E}\int_0^T \left(X_t^2\,dt + c\,d\left(\check{\xi}_*\right)_t\right) \leq \beta.$$

This gives the desired equality $Q(x;\xi_*) = \beta$, thanks to (3.19), and ξ_* is clearly an optimal admissible control. Indeed, we have actually obtained that

$$Q(x;\xi_*) = \lim_{T\to\infty} \frac{1}{T}\mathsf{E}\int_0^T \left(X_t^2\,dt + c\,d\left(\check{\xi}_*\right)_t\right) = \beta.$$

Finally we should give the proof of (3.19). We need the following lemma.

Lemma 1. *Suppose that $\phi(t), \varphi(t)$ are non-negative measurable functions defined on interval $[0,\infty)$. If $\varphi(t)$ is increasing and*

$$\liminf_{T\to\infty} \left[\frac{1}{aT}\phi(T) + \frac{1}{T}\left(\int_0^T \phi(t)\,dt + \varphi(T)\right)\right] \geq k$$

for some $a, k > 0$, then

$$\liminf_{T\to\infty} \frac{1}{T}\left(\int_0^T \phi(t)\,dt + \varphi(T)\right) \geq k.$$

Proof of lemma: Let $u(T) \triangleq \int_0^T \phi(t)\,dt$. Without loss of generality, we can assume that $u(T) < \infty$ for all $T > 0$. It follows that the function $u(t)$ is differentiable with derivative

$$u'(t) = \phi(t)$$

for almost every $t \geq 0$. By the assumption, for every $\varepsilon > 0$, there exist T_0 so that

$$\frac{1}{a}\phi(t) + u(t) + \varphi(t) \geq (k - \varepsilon)t$$

for all $t > T_0$. Multiplying both sides by ae^{at}, we have

$$\frac{d}{dt}\left(e^{at}u(t)\right) + ae^{at}\varphi(t) = e^{at}\phi(t) + ae^{at}u(t) + ae^{at}\varphi(t) \geq ae^{at}(k - \varepsilon)t$$

for almost every $t \geq T_0$. For any $T > T_0$, integrating on both sides from T_0 to T, and observing that φ is increasing by the assumption, we have

$$e^{aT}u(T) - e^{aT_0}u(T_0) + (e^{aT} - e^{aT_0})\varphi(T) \geq (k - \varepsilon)\left(Te^{aT} - T_0e^{aT_0} - \frac{1}{a}(e^{aT} - e^{aT_0})\right)$$

for every $T > T_0$. Dividing both sides by Te^{aT} and then letting $T \to \infty$, we have

$$\liminf_{T\to\infty} \frac{1}{T}\left(\int_0^T \phi(t)\,dt + \varphi(T)\right) \geq k - \varepsilon.$$

But ε is an arbitrary positive number. We complete the proof. $\sharp$

Now let $\phi(t) = \mathsf{E}X_t^2$, $\varphi(t) = c\mathsf{E}\check{\xi}_t$ and $a = \frac{1}{\epsilon_1}$, we obtain (3.19) from (3.18). $\square$

4 Connection between discount problem and ergodic problem

The discount problem is closely connected to the ergodic problem. In this section and this section only, we are going to denote by $v_\alpha(x)$ the value function of discount problem (1.3). The main result in the section is the following theorem.

Theorem 3. *The values for the discounted problem and the ergodic problem are connected in the following Abelian sense:*

$$\lim_{\alpha\to 0} \alpha v_\alpha(x) = \beta$$

for all $x \in \mathbb{R}$.

Proof: The method we use is very similar to the one adopted in [12], theorem 4. Let ξ_* be the optimal control policy, as defined in Theorem 2, for the ergodic problem (1.4). Write

$$F(T) = \mathsf{E}\int_0^T X_t^2\,dt + c\,d(\check{\xi}_*)_t, \qquad \text{where} \quad X_t \triangleq x + W_t + (\xi_*)_t,$$

for every $T > 0$. It follows from Remark 3 that

$$\lim_{t\to\infty} \frac{F(t)}{t} = \beta.$$

However, for any $\alpha > 0$, ξ_* is suboptimal for the discounted problem (1.3), i.e.

$$v_\alpha(x) \le J(x;\xi_*) = \int_0^\infty e^{-\alpha t}\, dF(t)$$

It follows from the Abelian theorem (see [22]) that

$$\limsup_{\alpha\to 0} \alpha v_\alpha(x) \le \beta. \tag{4.1}$$

It remains to establish the opposite inequality. Let ξ^* be the optimal control policy for the discount problem (1.3), as defined in Theorem 1. Abusing notation a bit, we still denote by $X = \{X_t; t \ge 0\}$ the state process, that is $X_t = x + W_t + \xi_t^*$. Applying the Doléan-Dade-Meyer formula to the process $\{e^{-\alpha t} V(X_t); t \ge 0\}$, we have

$$\mathsf{E} e^{-\alpha T} V(X_T) + \mathsf{E}\int_0^T e^{-\alpha t}\left(X_t^2\, dt + c\, d\check{\xi}_t^*\right) \ge \int_0^T \beta e^{-\alpha t}\, dt - \mathsf{E}\int_0^T \alpha e^{-\alpha t} V(X_t)\, dt,$$

using a similar argument in obtaining (3.16). By taking limit as $T \to \infty$, it follows that

$$\liminf_{T\to\infty} e^{-\alpha T}\mathsf{E} V(X_T) + v_\alpha(x) \ge \frac{\beta}{\alpha} - \mathsf{E}\int_0^\infty \alpha e^{-\alpha t} V(X_t)\, dt.$$

This implies, thanks to (3.15), that

$$\begin{aligned} \liminf_{T\to\infty} \epsilon_1 e^{-\alpha T}\mathsf{E} X_T^2 + v_\alpha(x) &\ge \frac{\beta}{\alpha} - \mathsf{E}\int_0^\infty \alpha e^{-\alpha t}\left(\epsilon_1 X_t^2 + \epsilon_2\right)\, dt \\ &\ge \frac{\beta}{\alpha} - \epsilon_1 \alpha v_\alpha(x) - \epsilon_2. \end{aligned}$$

However, it is easy to see that

$$\liminf_{T\to\infty} \mathsf{E} e^{-\alpha T} X_T^2 = 0,$$

otherwise $v_\alpha(x) = J(x;\xi^*) = \infty$, a contradiction. Therefore, we have

$$(1+\epsilon_1\alpha) v_\alpha(x) \ge \frac{\beta}{\alpha} - \epsilon_2.$$

Multiplying both sides by α, and observing $\lim_{\alpha\to 0}\alpha^2 v_\alpha(x) = 0$ by (4.1), we have

$$\liminf_{\alpha\to 0} \alpha v_\alpha(x) \ge \beta.$$

This complete the proof. □

5 Asymptotic Analysis as $\lambda \to \infty$

A relevant question for this formulation is the following: what is the cost of having such constraints on the intervention times? Or, what is the magnititude of the difference it makes from the scenerio where the control can be adjusted continuously and instantaneously? For the convenience of readers, a brief account of such control problems (where control policy ξ is only required to have bounded variation) is given below; see [12] for more details. It is well known that the optimal control processes are usually *singular* with respect to the Lebesgue measure for this type of problems.

5.1 Review of some singular control problems

Let $\mathcal{A}$ denote the set of all $\{\mathcal{F}_t\}$-adapted, left-continuous processes $\xi = \{\xi_t; t \geq 0\}$ such that for almost every $\omega \in \Omega$, the sample path $t \mapsto \xi_t(\omega)$ has bounded total variation on any compact interval on $[0, \infty)$, and $\xi_0(\omega) = 0$. The total variation process of ξ, still denoted by $\check{\xi}$, is given by $\check{\xi}_t = \int_0^t |d\xi_s|$, for every $t \geq 0$.

Suppose now $\mathcal{A}$ is the set of all admissible control processes, with corresponding state process

$$X_t = x + W_t + \xi_t, \qquad t \geq 0$$

for all $\xi \in \mathcal{A}$. The total expected discounted cost (with discount factor $\alpha > 0$) or the average cost per unit time are similarly defined as

$$J(x;\xi) \triangleq \mathsf{E} \int_0^\infty e^{-\alpha t} \left(X_t^2 \, dt + c \, d\check{\xi}_t \right); \tag{5.1}$$

$$Q(x;\xi) \triangleq \liminf_{T\to\infty} \frac{1}{T} \mathsf{E} \int_0^T \left(X_t^2 \, dt + c \, d\check{\xi}_t \right). \tag{5.2}$$

The objective is to minimize $J(x;\xi)$ or $Q(x;\xi)$ over all admissible controls $\xi \in \mathcal{A}$. The solutions to these two problems are described as follows (see [12] for more details).

Let $v_0(x)$ denote the value function of the discounted problem (5.1). It can be shown that $v_0(x)$ is the unique twice-continuously differentiable, even-symmetric function that satisfies the following variational inequality.

$$v_0'(0) = 0; \tag{5.3}$$

$$v_0'(x_0^*) = c; \tag{5.4}$$

$$\frac{1}{2} v_0''(x) - \alpha v_0(x) + x^2 = 0; \qquad 0 \leq x < x_0^* \tag{5.5}$$

$$v_0(x) - v_0(x_0^*) - c(x - x_0^*) = 0; \qquad x \geq x_0^* \tag{5.6}$$

where x_0^* is the unique positive number that satisfies the transcendental equation

$$0 = \frac{2}{\sqrt{2\alpha}} \coth(\sqrt{2\alpha} x_0^*) \left(x_0^* - \frac{c\alpha}{2} \right) - \frac{1}{\alpha}. \tag{5.7}$$

The value function $v_0(x)$ is given by

$$v_0(x) = \left\{ \begin{array}{cl} A_0 \cosh(\sqrt{2\alpha} x) + \frac{1}{\alpha} x^2 + \frac{1}{\alpha^2} & ; \text{ if } 0 \leq x \leq x_0^* \\ v_0(x_0^*) + c(x - x_0^*) & ; \text{ if } x > x_0^* \end{array} \right\}, \tag{5.8}$$

where

$$A_0 = -\frac{1}{\sinh(\sqrt{2\alpha} x_0^*)} \frac{2}{\alpha\sqrt{2\alpha}} \left(x_0^* - \frac{c\alpha}{2} \right). \tag{5.9}$$

The optimal control strategy, as reader may have already guessed, is the following: *do not act when the state process is in the interval* $[-x_0^*, x_0^*]$, *while on the (reflecting) boundaries* $\pm x_0^*$ *exert exact amount of control in order not to exit the interval. If we happen to start from outside the interval, then push the state process immediately to the nearest boundary (*$\pm x_0^*$ *respectively).* This optimal strategy, with the possible discontinuity at $t = 0$, is *singular* with respect to the Lebesgue measure.

The optimal control policy to the ergodic problem (5.2) is very similar, but with different reflecting boundaries $\pm b_0$. It can be shown that the b_0 satisfies the equation

$$\frac{4}{3} b_0^3 = c, \tag{5.10}$$

and the value of this ergodic control problem, denoted by β_0, is

$$\beta_0 = b_0^2 = \frac{1}{3} b_0^2 + \frac{c}{2b_0}. \tag{5.11}$$

5.2 Asymptotics

Here we study the asymptotics as λ, the intensity of the auxiliary Poisson process, goes to infinity (or the mean interarival time $h = \lambda^{-1}$ goes to zero). It is natural to expect that the value functions and the optimal exercise boundaries for problems (1.3) and (1.4) will approach those of the corresponding singular control problems that we have discussed in the previous subsection. In this section, we will denote by $v_h(x)$ the value function of the discounted problem (1.3) (with discounted factor α fixed), and by $\pm x_h^*$ its optimal threshholds. β_h and $\pm b_h$ serve similar purpose for the ergodic problem (1.4).

The following result says that the value functions converge with rate λ^{-1}. In other words, the cost of the constraint on the intervention times is of magnitude λ^{-1} provided that λ is big enough. The optimal exercise boundaries, however, converge with rate $\sqrt{\lambda^{-1}}$ and rate coefficient $-\frac{\sqrt{2}}{2}$, always.

Theorem 4. *Let* $h = \lambda^{-1}$ *be the mean interarrival time. The value function for the discounted problem* (1.3) *with discount factor* $\alpha > 0$ *satisfies*

$$\begin{aligned} v_h(x) &= v_0(x) + \left\{ \begin{array}{ccl} \frac{1}{2\alpha} \frac{\cosh(\sqrt{2\alpha}x)}{\cosh(\sqrt{2\alpha}x_0^*)} & ; & 0 \le x < x_0^* \\ \frac{1}{2\alpha} + (x - x_0^*)(x + x_0^* - \alpha c) & ; & x \ge x_0^* \end{array} \right\} \cdot h + O(h\sqrt{h}), \\ x_h^* &= x_0^* - \sqrt{\frac{h}{2}} + O(h). \end{aligned} \tag{5.12}$$

The value for the ergodic problem (1.4) *satisfies*

$$\beta_h = \beta_0 + \frac{1}{2}h + O(h\sqrt{h}), \tag{5.13}$$
$$b_h = b_0 - \sqrt{\frac{h}{2}} + O(h).$$

Here $(v_0(x), x_0^*)$ and (β_0, b_0) are the solutions to the singular discount and ergodic problems respectively, as discussed in the preceding subsection.

Proof: The proof is straightforward computation using the Implicit Function Theorem. From now on, let $\iota \triangleq \sqrt{h}$.

We first prove (5.13). It follows from (3.11) that b_h is the unique positive solution to the equation

$$0 = \frac{2}{3}b^2 + \iota^2 - \frac{c}{2b} + \sqrt{2}\iota b := g(b;\iota)$$

for all $h \geq 0$ (note the equation reduces to (5.10) as h=0). It is not difficult to see that the functions $g(\cdot,\iota)$ and $g(b,\cdot)$ are both increasing for $b > 0, \iota \geq 0$. Therefore, $b_h \uparrow b_0$ as $h \downarrow 0$ (or $\iota \downarrow 0$). However, for all $b > 0, \iota \geq 0$,

$$g_b = \frac{4}{3}b + \frac{c}{2b^2} + \sqrt{2}\iota \neq 0.$$

It follows from the Implicit Function Theorem that b_h is continuously differentiable (actually, smooth) with respective ι. In particular, thanks to (5.10), we have

$$\left.\frac{db_h}{d\iota}\right|_{\iota=0} = -\left.\frac{g_\iota}{g_b}\right|_{\iota=0,b=b_0} = -\frac{\sqrt{2}b_0}{\frac{4}{3}b_0 + \frac{c}{2b_0^2}} = -\frac{\sqrt{2}b_0}{\frac{4}{3}b_0 + \frac{2}{3}b_0} = -\frac{\sqrt{2}}{2}.$$

By (5.11) and (3.12), we have $\beta_h \downarrow \beta_0$ as $h \downarrow 0$. Furthermore,

$$\left.\frac{d\beta_h}{d\iota}\right|_{\iota=0} = \left(\frac{2}{3}b_h - \frac{c}{2b_h^2}\right)\cdot\left.\frac{db_h}{d\iota}\right|_{\iota=0} = \left(\frac{2}{3}b_0 - \frac{c}{2b_0^2}\right)\cdot\left(-\frac{\sqrt{2}}{2}\right) = 0$$

and

$$\left.\frac{d^2\beta_h}{d\iota^2}\right|_{\iota=0} = \left(\frac{2}{3}b_h - \frac{c}{2b_h^2}\right)\cdot\left.\frac{d^2b_h}{d\iota^2}\right|_{\iota=0} + \left(\frac{2}{3} + \frac{c}{b_h^3}\right)\cdot\left.\left(\frac{db_h}{d\iota}\right)^2\right|_{\iota=0} = 0 + \left(\frac{2}{3} + \frac{4}{3}\right)\cdot\frac{1}{2} = 1.$$

We complete the proof of (5.13).

The proof for (5.12) is similar. The optimal exercise boundary x_h^* is the unique positive solution to the equation

$$0 = \left(\frac{2\iota}{\sqrt{2(\alpha\iota^2+1)}} + \frac{2}{\sqrt{2\alpha}}\coth(\sqrt{2\alpha}x)\right)\left(x - \frac{c\alpha}{2}\right) - \frac{1}{\alpha(\alpha\iota^2+1)} := G(x;\iota)$$

for all $\iota \geq 0$ (note the equation reduces to (5.7) when $\iota = 0$). Indeed, $x_h^* > \frac{c\alpha}{2}$ for all $\iota \geq 0$. Since $G(\cdot;\iota)$ and $G(x;\cdot)$ are both increasing functions for $x \geq \frac{c\alpha}{2}$ and $\iota \geq 0$, we have $x_h^* \uparrow x_0^*$ as $h \downarrow 0$. It is not difficult to verify that

$$
\begin{aligned}
G_x\Big|_{\iota=0,x=x_0^*} &= \frac{2}{\sqrt{2\alpha}}\coth(\sqrt{2\alpha}x_0^*) - \frac{2}{\sinh^2(\sqrt{2\alpha}x_0^*)}\left(x_0^* - \frac{c\alpha}{2}\right) \\
&= \frac{2}{\sqrt{2\alpha}}\coth(\sqrt{2\alpha}x_0^*) - \frac{2}{\sinh^2(\sqrt{2\alpha}x_0^*)}\cdot\frac{1}{\sqrt{2\alpha}}\frac{\sinh(\sqrt{2\alpha}x_0^*)}{\cosh(\sqrt{2\alpha}x_0^*)} \\
&= \frac{2}{\sqrt{2\alpha}}\left(\frac{\cosh^2(\sqrt{2\alpha}x_0^*) - 1}{\sinh(\sqrt{2\alpha}x_0^*)\cosh(\sqrt{2\alpha}x_0^*)}\right) = \frac{2}{\sqrt{2\alpha}}\tanh(\sqrt{2\alpha}x_0^*) = 2\left(x_0^* - \frac{c\alpha}{2}\right) \neq 0.
\end{aligned}
$$

Here the second and the last equalities follow from (5.7). Therefore,

$$
\frac{dx_h^*}{d\iota}\Big|_{\iota=0} = -\frac{G_\iota}{G_x}\Big|_{\iota=0,x=x_0^*} = -\frac{\sqrt{2}\left(x_0^* - \frac{c\alpha}{2}\right)}{2\left(x_0^* - \frac{c\alpha}{2}\right)} = -\frac{\sqrt{2}}{2}.
$$

This proves the second part of (5.12). As for the first part, we first show that $v_h(x) \to v_0(x)$ as $h \to 0$ for all $x \geq 0$ (even symmetry yields the convergence for $x < 0$). We will denote by (A_h, B_h) the constants (A, B) in (2.9) for the obvious reason. Since $x_h^* \uparrow x_0^*$ as $h \to 0$, it follows that $A_h \to A_0$ (see (5.9)), $v_h(x_h^*) \to v_0(x_0^*)$ as $h \to 0$ and $v_h(x) \to v_0(x)$ for all $0 \leq x < x_0^*$. Noting $x_h^* < x_0^*$ for all $h > 0$, we have, for all $x \geq x_0^*$, that

$$
\begin{aligned}
\text{(5.14)} \qquad 0 \;&\leq\; B_h e^{-\sqrt{2(\alpha+\lambda)}x} = \frac{2}{(\alpha+\lambda)\sqrt{2(\alpha+\lambda)}}\left(x_h^* - \frac{c\alpha}{2}\right) e^{\sqrt{2(\alpha+\lambda)}(x_h^*-x)} \\
&\leq\; \frac{2\iota^3}{(\alpha\iota^2+1)\sqrt{2(\alpha\iota^2+1)}}\left(x_0^* - \frac{c\alpha}{2}\right) = O(\iota^3) \quad \to \quad 0,
\end{aligned}
$$

as $\iota \to 0$ (or $\lambda \to \infty$). Hence, $v_h(x) \to v_0(x)$ readily for $x \geq x_0^*$.

We will compute $\frac{dv_h(x)}{d\iota}$ and $\frac{d^2v_h(x)}{d\iota^2}$ for $0 \leq x < x_0^*$ and $x \geq x_0^*$ seperately. For $0 \leq x < x_0^*$, we have

$$
\frac{dv_h(x)}{d\iota} = \frac{dA_h}{d\iota}\cosh(\sqrt{2\alpha}x), \qquad \frac{d^2v_h(x)}{d\iota^2} = \frac{d^2A_h}{d\iota^2}\cosh(\sqrt{2\alpha}x).
$$

However, straightforward calculation shows that

$$
\text{(5.15)} \quad \frac{dA_h}{d\iota}\Big|_{\iota=0} = \left[-\frac{1}{\sinh(\sqrt{2\alpha}x_0^*)}\frac{2}{\alpha\sqrt{2\alpha}} + \frac{\cosh(\sqrt{2\alpha}x_0^*)}{\sinh^2(\sqrt{2\alpha}x_0^*)}\frac{2}{\alpha}\left(x_0^* - \frac{c\alpha}{2}\right)\right]\cdot\frac{dx_h^*}{d\iota} = 0 \qquad \text{(by (5.7))}
$$

and

$$
\begin{aligned}
\frac{d^2A_h}{d\iota^2}\Big|_{\iota=0} &= \left[\frac{4}{\alpha}\frac{\cosh(\sqrt{2\alpha}x_0^*)}{\sinh^2(\sqrt{2\alpha}x_0^*)} + \frac{2\sqrt{2\alpha}}{\alpha}\left(x_0^* - \frac{c\alpha}{2}\right)\frac{\sinh^2(\sqrt{2\alpha}x_0^*) - 2\cosh^2(\sqrt{2\alpha}x_0^*)}{\sinh^3(\sqrt{2\alpha}x_0^*)}\right]\cdot\left(\frac{dx_h^*}{d\iota}\right)^2 \\
\text{(5.16)} \qquad &= \left[\frac{4}{\alpha}\frac{\cosh(\sqrt{2\alpha}x_0^*)}{\sinh^2(\sqrt{2\alpha}x_0^*)} + \frac{2}{\alpha}\frac{\sinh^2(\sqrt{2\alpha}x_0^*) - 2\cosh^2(\sqrt{2\alpha}x_0^*)}{\cosh(\sqrt{2\alpha}x_0^*)\sinh^2(\sqrt{2\alpha}x_0^*)}\right]\cdot\frac{1}{2} \qquad \text{(by (5.7))} \\
&= \frac{1}{\alpha}\frac{1}{\cosh(\sqrt{2\alpha}x_0^*)}.
\end{aligned}
$$

Therefore, for $0 \le x < x_0^*$,

$$\frac{dv_h(x)}{d\iota} = 0, \qquad \frac{d^2 v_h(x)}{d\iota^2} = \frac{1}{\alpha}\frac{\cosh(\sqrt{2\alpha}x)}{\cosh(\sqrt{2\alpha}x_0^*)}.$$

For $x \ge x_0^*$, it follows from (2.7) and (5.14) that

$$\begin{aligned}
\left.\frac{dv_h(x)}{d\iota}\right|_{\iota=0} &= \left.\frac{d\big(v_h(x_h^*) - cx_h^*\big)}{d\iota}\right|_{\iota=0} \\
\left.\frac{d^2 v_h(x)}{d\iota^2}\right|_{\iota=0} &= 2x^2 - 2c\alpha x - 2\alpha\big(v_0(x_0^*) - cx_0^*\big) + \left.\frac{d^2\big(v_h(x_h^*) - cx_h^*\big)}{d\iota^2}\right|_{\iota=0}.
\end{aligned}$$

However, it follows from (2.6) that

$$v_h(x_h^*) = A_h \cosh(\sqrt{2\alpha}x_h^*) + \frac{1}{\alpha}(x_h^*)^2 + \frac{1}{\alpha^2}.$$

Therefore, we have

$$\left.\frac{d\big(v_h(x_h^*) - cx_h^*\big)}{d\iota}\right|_{\iota=0} = \left.\frac{dA_h}{d\iota}\right|_{\iota=0}\cosh(\sqrt{2\alpha}x_0^*) + \left(A_0\sqrt{2\alpha}\sinh(\sqrt{2\alpha}x_0^*) + \frac{2}{\alpha}x_0^* - c\right)\cdot\left.\frac{dx_h^*}{d\iota}\right|_{\iota=0} = 0$$

by (5.15), (5.9). Similarly,

$$\begin{aligned}
\left.\frac{d^2\big(v_h(x_h^*) - cx_h^*\big)}{d\iota^2}\right|_{\iota=0} &= \left.\frac{d^2 A_h}{d\iota^2}\right|_{\iota=0}\cosh(\sqrt{2\alpha}x_0^*) + \left(2\alpha A_0\cosh(\sqrt{2\alpha}x_0^*) + \frac{2}{\alpha}\right)\cdot\left(\left.\frac{dx_h^*}{d\iota}\right|_{\iota=0}\right)^2 \\
&= \frac{1}{\alpha}. \qquad \text{(by (5.16), (5.7) and (5.9))}
\end{aligned}$$

It follows then

$$\begin{aligned}
\left.\frac{d^2 v_h(x)}{d\iota^2}\right|_{\iota=0} &= 2x^2 - 2c\alpha x - 2\alpha\big(v_0(x_0^*) - cx_0^*\big) + \frac{1}{\alpha} \\
&= \frac{1}{\alpha} + 2x^2 - 2c\alpha x - 2\alpha\cdot\left(\frac{(x_0^*)^2}{\alpha} - cx_0^*\right) \qquad \text{(by (5.7), (5.9) and (5.8))} \\
&= \frac{1}{\alpha} + 2(x - x_0^*)(x + x_0^* - c\alpha).
\end{aligned}$$

This completes the proof. □

6 Summary

In this paper, we consider a class of control problems sharing the common feature that the decision maker cannot freely choose the intervention times. Indeed, it is only allowed to exert control at the arrival times of an independent, uncontrolled exogenous Poisson signal process. Explicit solutions are obtained for both the discounted problem and the ergodic problem. Also studied is the asymptotics of such control problems as the intensity of the Poisson process goes to infinity. We find that the cost of such constraints is of magnitude $\frac{1}{\lambda}$ for big λ.

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