An Elementary Linear Regulator Problem

Here we consider a very elementary optimization problem called the linear regulator problem which is applied to a large number of design problems in engineering. Suppose the state process $X(t)$ is governed by the following ODE

$$\frac{dX}{dt} = aX(t) + bU(t); \qquad t \geq 0.$$

with initial condition $X(0) = x_0$. Here a, b are arbitrary constants and $U(t)$ is a *control* process. The objective is to minimize the following cost criteria by judiciously choose the control strategy $U(t)$:

$$\min_{\{U(t);\; t\in[0,1]\}} \left(X^2(1) + \lambda \int_0^1 U^2(s)\, ds \right),$$

where $\lambda > 0$ is a given constant. Roughly speaking, the objective is to make the terminal value $X(1)$ as close to zero as possible while keeping the control effort as little as posible.

The method we are going to use is called *dynamic programming*. To this end, we first expand our problem to incorporate different initial conditions

$$V(t,x) \stackrel{\triangle}{=} \min_U \left(X^2(1) + \lambda \int_t^1 U^2(s)\, ds \right) \qquad \text{given that } X(t) = x.$$

for all $s \in [0,1]$ and $x \in \mathbb{R}$ (*the original problem corresponds to* $V(0, x_0)$). Here V is called *value function*. Note

$$V(1,x) = x^2.$$

Can we obtain the equation for $V(t,x)$ and the optimal control strategy for all t and all x?

Suppose now at time t, $X(t) = x$. Consider the following strategy: impose control u for a small time interval $[t, t+dt]$ and then switch to the corresponding optimal control strategy. This control strategy will pay $\lambda u^2\, dt$ on this small time interval and reach

$$X(t+\, dt) = dX(t) = \big(aX(t) + bU(t)\big)\, dt = (ax + bu)\, dt.$$

This strategy is always suboptimal no matter what the control u is, in the sense that

$$V(t,x) \leq \lambda u^2\, dt + V\big(t + dt, x + dX(t)\big)$$

However,

$$V\big(t+dt, x+dX(t)\big) = V(t,x) + V_t\, dt + V_x\, dX_t = V_t\, dt + V_x(ax+bu)\, dt.$$

Therefore, we have

$$V(t,x) \le V(t,x) + \big(\lambda u^2 + buV_x + V_t + axV_x\big)\ dt \qquad \Rightarrow \qquad \lambda u^2 + buV_x + V_t + axV_x \ge 0$$

for any control u. In other words,

$$0 \le \min_{u \in \mathbb{R}} \big(\lambda u^2 + buV_x + V_t + axV_x\big)$$

However, if u happen to be the optimal control, say u^*, we should have

$$V(t,x) = \lambda u^2\, dt + V\big(t + dt, x + dX(t)\big)$$

and

$$V(t,x) = V(t,x) + \big(\lambda(u^*)^2 + bu^*V_x + V_t + axV_x\big)\ dt$$

which implies that

$$\lambda(u^*)^2 + bu^*V_x + V_t + axV_x = 0,$$

or

$$0 = \lambda(u^*)^2 + bu^*V_x + V_t + axV_x = \min_{u \in \mathbb{R}} \big(\lambda u^2 + buV_x + V_t + axV_x\big)$$

Finding an optimal controm process is now reduced to finding the minimum of some function.

This leads to

$$u^* = -\frac{bV_x}{2\lambda}$$

and

$$0 = -\frac{b^2V_x^2}{4\lambda} + V_t + axV_x, \qquad V(1,x) = x^2.$$

This is a *partial differential equation* for value function V.

Guess a solution of form $V(t,x) = f(t)x^2$, then

$$u^* = -\frac{b}{\lambda} f(t)x$$

and

$$0 = -\frac{b^2}{\lambda} f^2(t)x^2 + f'(t)x^2 + 2af(t)x^2,$$

which implies that

$$-\frac{b^2}{\lambda} f^2(t) + 2af(t) + f'(t) = 0, \qquad f(1) = 1.$$

This is the so-called *Riccati equation.* This ODE is seperable and can be solved very easily. Indeed, we have

1. for $a = 0$: the solution

$$f(t) = \frac{1}{1 + \frac{b^2}{\lambda}(1-t)}$$

is an increasing function and nonegative for $t \le 1$, and

$$V(0,x_0) = f(0)x_0^2 = \frac{\lambda}{\lambda + b^2} x_0^2, \qquad u^* = -\frac{bx}{\lambda + b^2(1-t)}$$

2. for $a \neq 0$: the solution

$$f(t) = \frac{\frac{2a\lambda}{b^2}}{1 + \left(\frac{2a\lambda}{b^2} - 1\right) e^{2a(t-1)}}$$

is a nonnegative for $t \leq 1$, and

$$V(0, x_0) = f(0)x_0^2 = \frac{\frac{2a\lambda}{b^2}}{1 + \left(\frac{2a\lambda}{b^2} - 1\right) e^{-2a}} x_0^2, \qquad u^* = -\frac{b}{\lambda} \frac{\frac{2a\lambda}{b^2}}{1 + \left(\frac{2a\lambda}{b^2} - 1\right) e^{2a(t-1)}} x.$$

Note, for $a < \frac{b^2}{2\lambda}$, $f(t)$ is an increasing function and for $a > \frac{b^2}{2\lambda}$, $f(t)$ is a decreasing function. For $a_* = \frac{b^2}{2\lambda}$, $f(t) \equiv 1$. □

The above formal argument provides the formulas for value function $V(t, x)$ and also the optimal control process u^*. We shall give a *verification lemma* to finalize the result rigorously, that is to prove the solutions we have obtained above is really the optimal ones.

Verification Lemma: Verify rigorously that $V(0, x_0) = f(0)x_0^2$ and $U^*(t) = -\frac{b}{\lambda} f(t)X(t)$.

Proof: Let $v(t, x) = f(t)x^2$. We first show that $v(0, x_0) \leq V(0, x_0)$. For any control process $U(t)$, consider function $\phi(t) = v\big(t, X(t)\big)$. It follows that

$$\phi'(t) = v_t + v_x x' = v_t + v_x \big(aX(t) + bU(t)\big)$$

However, we know

$$\min_{u \in R} \left(\lambda u^2 + v_t + buv_x + axv_x\right) = 0,$$

hence

$$\phi'(t) = v_t + v_x \big(aX(t) + bU(t)\big) \geq -\lambda U^2(t),$$

which implies that

$$v(1, X(1)) - v(0, x_0) = \phi(1) - \phi(0) \geq -\lambda \int_0^1 U^2(t)\, dt \quad \Rightarrow \quad X^2(1) + \lambda \int_0^1 U^2(t)\, dt \geq v(0, x_0)$$

Hence $V(0, x_0) \geq v(0, x_0)$. Repeat the above procedure with $U(t) = U^*(t) = -\frac{b}{\lambda} f(t)X(t)$. We will see all the inequalities are indeed equalities, so that

$$v(0, x_0) = X^2(1) + \lambda \int_0^1 (U^*)^2(t)\, dt,$$

this implies that $V(0, x_0) = v(0, x_0)$ and $U^*(t)$ is an optimal control process. □