Introduction

1 Preliminaries

Limit: Consider a sequence $a_1, a_2, \cdots, a_n, \cdots$. We say the sequence *converges* to a, or

$$\lim_{n\to\infty} a_n = a \qquad (\text{sometimes} \quad \lim a_n = a),$$

if for any $\epsilon > 0$, there exists N such that

$$|a_n - a| < \epsilon; \qquad \forall\ n \geq N.$$

Similarly, we can define the limit of the following type

$$\lim_{x\to x_0} f(x)$$

where f is a function.

Example: Given a fixed number a, we have

$$\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a.$$

Example: We have

$$\lim_{x\to 0} x \log x = 0, \qquad \lim_{x\to+\infty} x^a e^{-bx} = 0 \quad (b > 0)$$

Example: What is the limit of the following sequence

$$\sqrt{2}, \quad \sqrt{2+\sqrt{2}}, \quad \sqrt{2+\sqrt{2+\sqrt{2}}}, \quad \cdots$$

More generally, can you determine for any given $b > 0$, the limit of sequence

$$\sqrt{b}, \quad \sqrt{b+\sqrt{b}}, \quad \sqrt{b+\sqrt{b+\sqrt{b}}}, \quad \cdots$$

Solution: The sequence can be written as

$$a_1 = b, \quad a_2 = \sqrt{b + a_1}, \quad a_3 = \sqrt{b + a_2}, \quad \cdots \quad , \quad a_{n+1} = \sqrt{b + a_n}, \quad \cdots$$

Suppose $\lim a_n = x$ (can you *prove* that the limit exists?). We have

$$x = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{b + a_n} = \sqrt{b + x} \quad \Rightarrow \quad x^2 - x - b = 0.$$

This yields that

$$x = \frac{1+\sqrt{1+4b}}{2} \quad \text{or} \quad x = \frac{1-\sqrt{1+4b}}{2} < 0.$$

But x cannot be negative, hence

$$x = \lim a_n = \frac{1+\sqrt{1+4b}}{2}.$$

Diffenetiation: Suppose $y = f(x)$ is a function. We say f is *differentiable* at x if the following limit exists:

$$\lim_{\triangle x \to 0} \frac{f(x+\triangle x) - f(x)}{\triangle x}$$

The limit is called the *(first order) derivative.* It is often denoted by $y', f', \dot{y}$ or most informatively $\frac{dy}{dx}$ (or $\frac{df}{dx}$). Second order derivative is defined as the derivative of the first order derivative, it is often denoted by $y'', f'', \ddot{y}$ or $\frac{d^2y}{dx^2}$ (or $\frac{d^2f}{dx^2}$). More generally, the n-th derivative is denoted by $y^{(n)}, f^{(n)}$.

Differentiation and Integration: We have

$$f(b) - f(a) = \int_a^b f'(x)\, dx.$$

Some useful rules: 1. *Chain rule for differentiation:* Suppose $y = f\big(g(x)\big)$. We have

$$\frac{dy}{dx} = f'\big(g(x)\big) \cdot g'(x)$$

2. *Integration by parts:*

$$\int u\, dv + \int v\, du = uv$$

or

$$\int_a^b f(x)g'(x)\, dx = f(x)g(x)\Big|_{x=a}^{x=b} - \int_a^b f'(x)g(x)\, dx$$

2 Introduction to ODE

Ordinary Differential Equations (ODE) are often encountered in physics, economics, engineering and social sciences, etc. The unknown in an ordinary differential equation is a real *function*, say

$$y = f(x); \qquad x \in (a, b)$$

which satisfies the equation (of *order* n)

$$F\left(x; f(x), f'(x), \cdots, f^{(n)}(x)\right) = 0; \qquad \forall x \in (a, b).$$

Sometime it is convenient to adopt notation without ambiguity

$$F\left(x; y, y', \cdots, y^{(n)}\right) = 0; \qquad \forall x \in (a, b).$$

Terminologies: 1. The above ordinary differential equation is said to be **linear** if F is a linear function of the variables $y, y', \cdots, y^{(n)}$ for every fixed x. A *linear* ODE shall assume the following form:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

It is said to be **non-linear** otherwise.

2. n is the **order** of the differential equation, which is the order of the highest derivative involved in the equation.

Exercise: Suppose $y_1(x)$ and $y_2(x)$ both satisfy the linear differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$$

on some interval I. Show that, for any constant c_1, c_2, function

$$y(x) \stackrel{\triangle}{=} c_1 y_1(x) + c_2 y(x)$$

is also a solution of the above differential equation.

Proof: Trivial. □

Notation: We shall use y' or $\dot{y}$ (resp. y'' or $\ddot{y}$) to denote $\frac{dy}{dx}$ (resp. $\frac{d^2y}{dx^2}$), etc.

The following are several ordinary differential equations

$$\begin{array}{ll} \ddot{y} + 5\sin y = -3x & (\textit{nonlinear}) \\ \ddot{y} \cdot [e^y + 2x] = 4\sqrt{|\cos x|} & (\textit{nonlinear}) \\ 5x\ddot{y} + \sin x \cdot \dot{y} = x^2 & (\textit{linear}) \end{array}$$

To keep in mind, a lot of (or, most) ordinary differential equations do not have solutions that can be represented by elemetary functions. Therefore it will be important to study the ordinary differential equation qualitatively (say, existence, uniqueness, asympotics and stability) and also numerically. Interested students should have no trouble finding litertures on these subjects.

3 Examples

Here we present several applications of ordinary differential equations.

Simple Interest Rate: Suppose you deposit a dollar money to a saving account at time $t = 0$. The saving account produces a simple interest (interest does *not* generate interest) with rate r per year. For any time $t = T$, find out the amount of money in your saving account.

Solution: Let $y = f(t)$ denote the total amount of money in your saving account at time t. we have

$$dy = ardt, \quad \forall t \geq 0 \qquad \Longleftrightarrow \qquad \dot{y} = ar$$

which has solutions of form

$$y = f(t) = art + c; \qquad \forall t \geq 0.$$

Here c is a constant to be determined. However, observing the initial condition

$$f(0) = a$$

we obtain $c = a$. Hence $f(t) = art + a = a(1 + rt)$. □

Compound Interest Rate: Redo the previous example assuming the saving account produces a compound interest (interest does generate interest) with r per year. If you want to have 1 million dollars in you account after 20 years, how much should you deposit if $r = 5\%$?

Solution: Now the differential equation will take form

$$dy = yr dt, \quad \forall t \geq 0 \qquad \Longleftrightarrow \qquad \dot{y} = ry \qquad \Longleftrightarrow \frac{d \log y}{dt} \equiv r$$

which has solutions of form

$$\log |y| = \log f(t) = rt + c, \qquad \forall t \geq 0 \quad \Rightarrow \quad y = f(t) = c \cdot e^{rt}$$

for some constant c. But $f(0) = a$, hence $c = a$, which implies

$$f(t) = ae^{rt}, \qquad \forall t \geq 0$$

If $r = 5\%$, $f(20) = 1$(million), we have to put

$$a = \frac{f(20)}{e^{0.05 \cdot 20}} = \frac{1}{e}(\text{million}) = 0.368(\text{million})$$

right now. $\square$

Radioactive decay; carbon dating: American chemist *Willard Libby* (1908-1980) developed the method of *carbon dating* in the early 1950's, which won him the Nobel Prize for chemistry in 1960. The essential idea is as follows.

(1) all living organisms contain two isotopes of carbon, namely, C^{12} and C^{14}. See the following graph.

(2) The element C^{12} is stable, while C^{14} is radioactive.

(3) The ratios of C^{12} to C^{14} in any the *living* organisms remains *constant*.

(4) After the organism dies, C^{14} begins to decay because of radiation and is no longer replaced. But C^{12} remains stable.

(5) The dacay of C^{14} is governed by the following formula. Let $y = f(t)$ is the total mass of C^{14} at time t. We have

$$\frac{dy}{dt} = -ky,$$

for some constant k. Just like the preceding compound interest example, we have

$$y = f(t) = ae^{-kt}$$

for some constant a. In this case, $a = f(0)$, the total mass of C^{14} at the organism's death.

The rate k can be expressed in terms of **half-life** T, the time required for any amount of mass a to be reduced by half, to $\frac{a}{2}$. The the above equation gives

$$\frac{a}{2} = ae^{-rT} \quad \Rightarrow \quad k = \frac{\log 2}{T}.$$

For C^{14}, half-life $T = 5570$ years, and $k = 0.0001244$.

(6) Suppose we can determine $f(0)$ (total mass of C^{14} at death) and y (the remaining mass of C^{14}), we can calculate t, the time when the organism died.

An actual historical event: In 1940, a group of boys was hiking in the vicinity of a town in France named Lascaux. They suddenly became aware that their dog had disappeared. In the following search the dog was found in a deep hole from which he could not climb out. When one boy lowered himself into the hole for the rescue, he made a discovery that the hole was once a part of the roof of an ancient cave that had covered with brush. On the walls of the cave there are marvellous paintings of wild horses, cattle, and a fierce-looking black beast which resembled our bull. This accidental discovery created a sensation. Besides the wall painting and other articles of archaeological interest, there were also the charcoal remains of a fire. The question is: determine from the charcoal remains how long ago the cave dwellers lived.

By a chemical analysis of the charcoal (from the rate of β-particle emission), one can determine the mass of C^{14} present at the time of discovery. However, chemical analysis also yield the mass of C^{12} in the charcoal, which remained same in time. From (3) one can then proceed to determine the mass of C^{14} at the time when the wood was used by the cave dwellers. It was found out the 85.5% of C^{14} was lost, or 14.5% of C^{14} remained. Therefore, we have

$$0.145a = ae^{-kt} \quad \Rightarrow \quad t = \frac{\log 0.145}{-k} = \frac{\log 0.145}{-0.0001244} = 15,523.$$

Therefore, the cave dweller lived about 15,500 years ago.

Law of Motion: The famous Newton's Law of Motion: *"product of mass and acceleration equals external force"*. In another word,

$$F = ma,$$

where F is external force, m is mass and a is acceleration.

Maximum Altitude: If we throw a ball straight up with velocity v, how high will it go? Here we assume there is no resistance due to air or wind.

Solution: Let us take upward as the positive direction. Suppose $x(t)$ is the height of the ball at time t. Then the velocity and acceleration of the ball at time t are respectively $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$. However, since the only external force the ball bear is its own weight (or, *gravitational attraction*), which is downwards with magnitude.

$$w = mg,$$

where $g \approx 32$ feet/sec^2 is the *acceleration due to gravity*. Hence, by Law of Motion, we obtain

$$\frac{d^2x}{dt^2} = -g. \qquad \text{“}-\text{” sign is due to the downward direction of gravity}$$

with initial conditions $x(0) = 0$, $\frac{dx}{dt}\big|_{t=0} = v$. The differential equation gives

$$\frac{dx}{dt} = -gt + c, \qquad \text{for some constant } c.$$

This in turn yields

$$x(t) = -\frac{1}{2}gt^2 + ct + K, \qquad \text{for some constants } c,\, K.$$

However, the initial condition gives $K \equiv 0$ and $c \equiv v$. Hence

$$x(t) = -\frac{1}{2}gt^2 + vt = -\frac{1}{2}g \cdot \left(t - \frac{v}{g}\right)^2 + \frac{v^2}{2g}.$$

The maximum of x (or, altitude) of the ball is achieved at time $t = \frac{v}{g}$, and

$$x_{max} = \frac{v^2}{2g}.$$

Exercise: Suppose a ball is dropped from a height h. How long will it take to reach the ground?

Solution: We have differential condition

$$\frac{d^2x}{dt^2} = -g$$

with initial condition $x(0) = h$, $\frac{dx}{dt}\big|_{t=0} = 0$. Its solution is

$$x(t) = -\frac{1}{2}gt^2 + h.$$

The ball takes time t^* to reach the ground, where t^* is given by

$$x(t^*) = 0 \quad \Rightarrow \quad t^* = \sqrt{\frac{2h}{g}}.$$

Maximum Altitude (Method 2): The above example actually gives us more than asked – we also obtain the time to the maximum altitude. In the following, let

$$f(v) \stackrel{\triangle}{=} \text{The maximum altitude a ball can reach, with initial speed } v$$

be as a function of initial speed.

Suppose the initial speed is v, after a small time-increment dt, the ball will travel up altitude $v\,dt$. However, its speed will reduce to $v - g\,dt$. This yields

$$f(v) - f(v - g\,dt) = v\,dt \;\Rightarrow\; \frac{f(v - g\,dt) - f(v)}{-g\,dt} = \frac{v}{g} \;\Rightarrow\; f'(v) = \frac{v}{g}$$

Hence $f(v) = \frac{1}{2}\frac{v^2}{g} + c$ for some constant c. However, $f(0) = 0 \Rightarrow c = 0$. Hence

$$\text{Maximum altitude with initial speed } v \equiv \frac{v^2}{2g}$$

Law of Conservation of Energy: In this example, we will prove a baby law of conservation of energy. Consider a perfectly smooth slide; see the following graph. A kid slides down from the top. Suppose at time t, his speed is v and altitude h. Prove that

$$\textbf{kinetic energy} + \textbf{potential energy} \stackrel{\triangle}{=} \frac{1}{2}mv^2 + mgh \equiv \text{const.}$$

where m is the kid's mass.

Proof: Suppose at time t, the kid is at position $(x(t), h(t))$ with speed $v(t)$. We have

$$v \cdot \sin\omega = -\frac{dh}{dt}.$$

It is clear that the kid is only subject to two forces: the gravity mg and the support from the silde whose direction is orthogonal to the slide. Since his acceleration is parallel to the slide, by Newton's Law, the total force he is subject to, say F, is also parallel to the slide. Hence, $F = mg \cdot \sin\omega$. However, $F = ma$ where $a = \frac{dv}{dt}$ is the acceleration. This yields

$$m\frac{dv}{dt} = mg\sin\omega \;\Rightarrow\; mv\frac{dv}{dt} = mgv\sin\omega = -mg\frac{dh}{dt} \;\Rightarrow\; \frac{d}{dt}\left(\frac{1}{2}mv^2 + mgh\right) = 0$$

In another word,

$$\frac{1}{2}mv^2 + mgh \equiv \text{const.}$$

Remark: The law of conservation of energy is a universal law. Interested student can try to prove the same arguement when the slide is replaced by any curve.

Proof: The kid is subject to the garvity and the support from the curve, which is vertical to the speed (tangent line). Let F be the total force, and we obtain, by trigonometry,

$$\frac{F}{\sin\alpha} = \frac{mg}{\sin(\pi - \alpha - \theta)} \;\Rightarrow\; F = mg\frac{\sin\alpha}{\sin(\alpha+\theta)},$$

which yields that

$$\ddot{x} \stackrel{\triangle}{=} \frac{d^2x}{dt^2} = F\sin\theta; \qquad \ddot{h} \stackrel{\triangle}{=} \frac{d^2h}{dt^2} = -F\cos\theta.$$

It is also clear that

$$\dot{h} \stackrel{\triangle}{=} \frac{dh}{dt} = -v\sin\alpha; \qquad \dot{x} \stackrel{\triangle}{=} \frac{dx}{dt} = v\cos\alpha.$$

Since $v^2 = (\dot{h})^2 + (\dot{x})^2$, it follows that

$$\frac{d}{dt}\left(\frac{1}{2}mv^2 + mgh\right) = \dot{h}\ddot{h} + \dot{x}\ddot{x} + mg\dot{h} = Fv\sin(\alpha+\theta) - mgv\sin\alpha = 0$$

This completes the proof. $\square$.

Variational Calculus: Why straight line is shortest? Two points, without loss generality, $(0,0)$ and (a,b). Find the curve of minimum length connecting these two points.

Solution: By a curve connecting these two points, we mean function

$$f : [0,a] \mapsto \mathbb{R}; \qquad f(0) = 0, \quad f(a) = b.$$

The length of the curve is given by

$$L(f) \equiv \int_0^a \sqrt{1+\dot{f}^2}dx; \qquad \text{where } \dot{f} \equiv \frac{df}{dx}.$$

Then the problem is asking to solve the following optimization problem

$$\inf_{f:\ [0,a]\mapsto\mathbb{R}} L(f) = \inf_{f:\ [0,a]\mapsto\mathbb{R}} \int_0^a \sqrt{1+\dot{f}^2}dx.$$

Suppose f^* is the optimal connecting curve. Let $g: \ [0,a] \mapsto \mathbb{R}$ be *any* function such that $g(0) = g(a) = 0$. For any real number ϵ, we know

$$f_\epsilon(x) \stackrel{\triangle}{=} f^*(x) + \epsilon g(x) \qquad (\text{Note: } f_0 \equiv f^*)$$

is another curve connecting $(0,0)$ and (a,b). But since f^* is the optimal curve, we have

$$l(\epsilon) \stackrel{\triangle}{=} L(f_\epsilon) = \int_0^a \sqrt{1+(\dot{f}^* + \epsilon\dot{g})^2}dx \quad \geq \quad L(f_0) = l(0).$$

Therefore, the funciton l *achieves its minimum at* $\epsilon = 0$. By Roll Theorem,

$$\left.\frac{dl}{d\epsilon}\right|_{ep=0} = 0$$

But (to simplify notation, we shall use f as f^* in the following)

$$
\begin{aligned}
\left.\frac{dl}{d\epsilon}\right|_{ep=0} &= \int_0^a \left.\frac{d}{d\epsilon}\sqrt{1+(\dot{f}+\epsilon\dot{g})^2}\right|_{\epsilon=0} dx \\
&= \int_0^a \frac{\dot{f}}{\sqrt{1+\dot{f}^2}}\dot{g}dx \\
&= -\int_0^a \frac{d}{dx}\left(\frac{\dot{f}}{\sqrt{1+\dot{f}^2}}\right)\cdot g\, dx. \equiv 0
\end{aligned}
$$

Here the last equality follows from intergration by parts and $g(0) = g(a) = 0$.

Since the above equality holds for *any* function g, it follows that

$$
\frac{d}{dx}\left(\frac{\dot{f}}{\sqrt{1+\dot{f}^2}}\right) \equiv 0 \quad \Rightarrow \quad \frac{\dot{f}}{\sqrt{1+\dot{f}^2}} = c
$$

for some constant c which is yet to be determined. This implies that $\dot{f}$ is also a constant. In another word, the shortest line connects two points is the *straight* line.

Variational Calculus: Brachistochrone Problem This is a more difficult problem, but the main idea is same as the previous example. The question is to find a curve along which a particle will slide without friction in the minimum time from one given point P to another Q, the second beinf lower than the first, but not directly beneath it. See the following graph.

Suppose the optimal curve is $y = f(x)$, with $f(0) = 0$ and $f(a) = b$. Then for any function $\phi : [0, a] \to \mathbb{R}$ such that $\phi(0) = \phi(a) = 0$ and any small number ϵ, $f_\epsilon \stackrel{\triangle}{=} f + \epsilon\phi$ would be another curve connecting points P and Q. How much time the particle need to take to reach Q along curve f_ϵ? By law of conservation of energy, its speed at point with altitude y from P is given by

$$
\frac{1}{2}mv^2 = mgy \;\Rightarrow\; v = \sqrt{gy}
$$

Hence the total time it needs is

$$
\begin{aligned}
t(\epsilon) &= \int_0^a \frac{\sqrt{(dx)^2+(dy)^2}}{v} = \int_0^a \sqrt{\frac{1+(\dot{y})^2}{gy}}\, dx \\
&= \int_0^a \sqrt{\frac{1+(\dot{f}+\epsilon\dot{\phi})^2}{g(f+\epsilon\phi)}}\, dx.
\end{aligned}
$$

Since function $t(\epsilon)$ achieve its minimum at $\epsilon = 0$, which yields $t'(0) = 0$. This yields

$$\begin{aligned} t'(0) &= \int_0^a \left(\frac{d}{d\epsilon} \sqrt{\frac{1 + (\dot{f} + \epsilon\dot{\phi})^2}{g(f + \epsilon\phi)}} \Bigg|_{\epsilon=0} \right) dx \\ &= \int_0^a \left(\frac{\dot{f}}{\sqrt{f(1+\dot{f}^2)}} \dot{\phi} - \phi \frac{1}{2} \sqrt{\frac{1+\dot{f}^2}{f^3}} \right) dx \\ &= \int_0^a \left(-\frac{d}{dx} \frac{\dot{f}}{\sqrt{f(1+\dot{f}^2)}} - \frac{1}{2} \sqrt{\frac{1+\dot{f}^2}{f^3}} \right) \cdot \phi \, dx \equiv 0. \end{aligned}$$

Here the second-to-last equality follows from integration by parts.

$$-\frac{d}{dx} \frac{\dot{f}}{\sqrt{f(1+\dot{f}^2)}} - \frac{1}{2} \sqrt{\frac{1+\dot{f}^2}{f^3}} \equiv 0$$

which,by algebra, yields

$$2f\ddot{f} + \dot{f} + 1 \equiv 0 \;\Rightarrow\; \dot{f}(2f\ddot{f} + \dot{f} + 1) = \frac{d}{dx}\left(f(1+\dot{f}^2) \right) \equiv 0.$$

Therefore

$$f(1+\dot{f}^2) \equiv \text{const.} = (\text{say})k^2.$$

Introduce a new variable: for $\theta \in [0, \pi]$,

$$f = k^2 \sin^2\theta \;\Rightarrow\; \dot{f} = \frac{df}{dx} = k^2 \cdot 2\sin\theta\cos\theta \frac{d\theta}{dx},$$

we obtain, observing $\dot{f} \geq 0$,

$$\dot{f} = \sqrt{\frac{k^2}{y} - 1} = \frac{\cos\theta}{\sin\theta} \;\Rightarrow\; \frac{dx}{d\theta} = 2k^2 \sin^2\theta$$

Solving this ODE, we get the parametric equations

$$x = k^2 \left(\theta - \frac{1}{2}\sin(2\theta) \right), \qquad y = k^2 \sin^2\theta.$$

To determine k, we only need to observe $f(a) = b$, and solve the equation system

$$a = k^2 \left(\theta - \frac{1}{2}\sin(2\theta) \right), \qquad b = k^2 \sin^2\theta,$$

which will yields the value of k (and also the θ corresponding to point Q). This curve is called **cycloid**.

Remark: An interesting phenomenon is that when $\frac{a}{b} > \frac{\pi}{2}$, the **cycloid** will actually go below point Q and then shoot upward to reach Q. $\square$