Applied Math 33

Second Exam, 6 November, 2001.

Points are as indicated. Please identify in a clear manner work that you want considered for credit. You may use your text book and one (1) calculus book for reference. Calculators that are capable of solving the calculus problems are not allowed.

1. (a) (18 pts.) Find the general solution to each of the following differential equations

$$y'' + y' - 6y = xe^x,$$

$$y'' + y' - 6y = e^{2x}.$$

   (b) (12 pts.) Solve the initial value problem for the second equation with the data $y(0) = 1, y'(0) = 0$.

Solution 1:

(a)

$$r^2 + r - 6 = 0 \Rightarrow (r+3)(r-2) = 0 \Rightarrow r = 2, -3.$$

Thus the solution to homogeneous equation is

$$c_1 e^{2x} + c_2 e^{-3x}.$$

(ai) Try $y_p = (ax+b)e^x$. Then $y_p^{'} = ae^x + (ax+b)e^x$ and $y_p^{''} = 2ae^x + (ax+b)e^x$. Therefore

$$2ae^x + (ax+b)e^x + ae^x + (ax+b)e^x - 6(ax+b)e^x = xe^x.$$

Thus

$$-4a = 1, 3a - 4b = 0 \Rightarrow a = -\frac{1}{4}, b = \frac{3}{4}a = \frac{-3}{16}.$$

(aii) Try axe^{2x} (Note that e^{2x} is a solution to the homogenous equation!). Then $y_p^{'} = ae^{2x} + 2axe^{2x}, y_p^{''} = 4ae^{2x} + 4axe^{2x}$. Thus we need

$$4ae^{2x} + axe^{2x} + ae^{2x} + 2axe^{2x} - 6axe^{2x} = e^{2x},$$

and so $5a = 1 \Rightarrow a = 1/5$.

(b) The general solution is

$$y = c_1 e^{2x} + c_2 e^{-3x} + \frac{1}{5}xe^{2x}.$$

The initial conditions imply $y(0) = 1 \Rightarrow c_1 + c_2 = 1$ and $y^{'}(0) = 0 \Rightarrow 2c_1 - 3c_2 + \frac{1}{5} = 0$, and Gaussian elimination gives $c_2 = \frac{11}{25}$ and $c_1 = \frac{14}{25}$.

2. A certain spring-mass system with damping has mass $m = 1$. If the periodic forcing term $\cos t$ is applied, then the position $u(t)$ of the mass satisfies

$$u'' + cu' + ku = \cos t.$$

(a) (15 pts.) Suppose that $u_p(t) = 2\sin t$ is observed to be a particular solution to the differential equation. Find c and k.

(b) (15 pts.) Find the general solution to the differential equation.

Solution 2: Inserting the particular solution gives

$$-2\sin t + 2c\cos t + 2k\sin t = \cos t,$$

which leads to the two equations $-2 + 2k = 0$ and $2c = 1$. Thus $c = \frac{1}{2}, k = 1$. The characteristic polynomial is then

$$r^2 + \frac{1}{2}r + 1 = 0$$

and thus

$$r = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - 4}}{2} = \frac{-\frac{1}{2} \pm \sqrt{\frac{15}{16}}i}{2} = \frac{-1}{4} \pm \sqrt{\frac{15}{64}}i.$$

The general solution is then

$$u(t) = c_1 c^{\frac{-1}{4}t} \cos\sqrt{\frac{15}{16}}t + c_2 c^{\frac{-1}{4}t} \sin\sqrt{\frac{15}{64}}t.$$

3. Consider the differential equation

$$(\sin x)y'' - (\cos x)y' = -1$$

for $0 < x < \pi$.

(a) (5 pts.) Show that both $\sin x$ and $1 + \sin x$ are solutions to this equation.

(b) (10 pts.) Construct a non-zero solution to the corresponding homogeneous equation. (Hint: There is not much work to do.)

(c) (10 pts.) Compute a second linearly independent solution to the homogeneous equation.

(d) (10 pts.) *Show* that the two solutions to the homogeneous equation are independent on the given interval.

(e) (5 pts.) Write down the general solution to the ODE, and solve the IVP with $y(0) = 0, y'(0) = 1$.

(Note: The integral of $\cot x$ on $0 < x < \pi$ is $\ln(\sin x)$.)

Solution 3:

(a)

$$\sin x(-\sin x) - (\cos x)\cos x = -\sin^2 x - \cos^2 x = -1$$

(b) For any two particular solutions $y_{p_1} - y_{p_2}$ is a solution to the homogenous equation, and thus one is $y_1(x) = 1$.

(c)

$$y_2 = u \cdot 1$$

$$(\sin x)u'' - (\cos x)u' = 0$$

$$\Rightarrow (\sin x)v' - (\cos x)v = 0$$

$$\Rightarrow \frac{dv}{v} = \frac{\cos x}{\sin x} = \cot x$$

$$\Rightarrow \ln v = \ln(\sin x)$$

$$\Rightarrow v = \sin x$$

$$\Rightarrow u = -\cos x$$

$$\Rightarrow y_2 = \cos x.$$

(d)

$$W(y_1, y_2)(x) = \begin{vmatrix} 1 & \cos x \\ 0 & -\sin x \end{vmatrix} = -\sin x.$$

Since this is nonzero on the interval independence follows.

(e)

$$y(x) = c_1 + c_2 \cos x + \sin x$$

$$y\left(\frac{\pi}{2}\right) = 0 \Rightarrow c_1 + 1 = 0$$

$$y'\left(\frac{\pi}{2}\right) = 1 \Rightarrow -c_2 = 1$$

$$\Rightarrow y(x) = -1 - \cos x + \sin x.$$