



Figure 1: graph of  $f(y)$  in the first question

### Fall 2001, AM33 Solution to the practice exam

1. Find all critical points of the differential equation

$$\frac{dy}{dt} = 4y - y^3$$

Classify each critical point as stable, unstable or semistable.

**Solution** The critical points are the roots of  $\frac{dy}{dt}$ . In this case:

$$\frac{dy}{dt} = f(y) = 4y - y^3 = y(2 - y)(2 + y)$$

Hence the critical points are :  $0, -2, +2$ . To decide stability it is useful to draw a graph.  $0$  is unstable because  $f(y)$  is negative (hence  $y$  is decreasing) when  $y$  is close to  $0$  from the left, and  $f(y)$  is positive (hence  $y$  is increasing) when  $y$  is close to  $0$  from the right. Thus  $0$  repels  $y$  when  $y$  is close to  $0$ . A similar argument says that  $-2$  and  $2$  are stable.

Let  $y(t)$  be the solution to this differential equation with  $y(0) = 1$ . Without solving the differential equations, state whether  $y(t)$  is increasing or decreasing as  $t$  increases, and find the limit of  $y(t)$  as  $t \rightarrow \infty$

**Solution** Looking at the graph we see that at  $y = 1$   $dy/dt > 0$  hence  $y$  is increasing. As  $t \rightarrow \infty$  it continues to increase towards the equilibrium point  $y = 2$ .

2. Solve the initial value problem:

$$\frac{dy}{dx} = \frac{-\sqrt{x+y}}{\sqrt{x+y}+2}, \quad y(0) = 1$$

**Solution** Do the following change of variables:

$$u = \sqrt{x+y}$$

By the chain rule we have:

$$\frac{du}{dx} = \frac{1}{2\sqrt{x+y}} \left(1 + \frac{dy}{dx}\right) = \frac{1}{u(u+2)}$$

This is a separable equation and has the implicit solution:

$$u^3/3 + u^2 = x + c$$

Substitute  $\sqrt{x+y}$  for  $u$ :

$$\begin{aligned} \frac{(x+y)\sqrt{x+y}}{3} + x + y &= x + c \\ \frac{(x+y)\sqrt{x+y}}{3} + y &= c \end{aligned}$$

The initial condition says  $y(0) = 1$ , we substitute that in the above display to find  $c$ :

$$1/3 + 1 = c \rightarrow c = 4/3$$

Thus we have the implicit solution:

$$\frac{(x+y)\sqrt{x+y}}{3} + y = 4/3$$

3. Consider the differential equation

$$y + e^x + (e^{3y-x} + 1)y' = 0$$

- (a) Find an integrating factor of the form  $\mu = \mu(x)$  or  $\mu = \mu(y)$  that turns this equation into an exact equation.

**Solution**

$$\frac{M_y - N_x}{N} = 1$$

Hence we know that there is an integrating factor of the form  $\mu = \mu(x)$  and it satisfies:

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N} = \mu(x)$$

(To see why this is the case refer to your book, page 94)

From this last DE we get  $\mu(x) = e^x$ .

- (b) Find an implicit relationship that solutions of this ODE satisfy.

**Solution** Multiplying the original DE with the integrating factor  $\mu(x)$  we get an exact equation:

$$\underbrace{e^x y + e^{2x}}_{\frac{\partial \psi}{\partial x}} + \underbrace{e^{3y} + e^x}_{\frac{\partial \psi}{\partial y}} y' = 0$$

Integrating the first part of this expression with respect to  $x$  we get:

$$\psi(x, y) = ye^x + \frac{e^{2x}}{2} + h(y)$$

where  $h(y)$  is an arbitrary function of  $y$ . From the DE above we know the partial derivative of  $\psi$  with respect to  $y$ . So we have:

$$\frac{\partial \psi}{\partial y} = e^x + h'(y) = e^{3y} + e^x$$

$$h'(y) = e^{3y} \rightarrow h(y) = \frac{e^{3y}}{3}$$

So we have the following implicit relation:

$$\psi(x, y) = ye^x + \frac{e^{2x}}{2} + \frac{e^{3y}}{3} = c$$

where  $c$  is a constant.

4. Consider the ODE

$$y' = \frac{x}{1+y}$$

(a) Is this ODE linear?

**Solution**

$$y' = \frac{x}{1+y} = f(x, y)$$

An ODE is linear when it is of the form  $y' + g(x)y = f(x)$ . The above DE is not of this kind.

(b) Identify the set of points in the  $(x, y)$ -plane where the existence and uniqueness theorem fails to hold.

**Solution** The existence and uniqueness theorem for the non-linear case is a local result and requires that  $f(x, y)$  and  $\frac{\partial f}{\partial y}$  be continuous in some open neighborhood of the initial point.  $f(x, y)$  is not continuous when  $y = -1$  so the existence and uniqueness theorem fails for  $y = -1$ .  $\frac{\partial f}{\partial y} = -x/(1+y)^2$ , so  $\frac{\partial f}{\partial y}$  is discontinuous at the same set of points, that is  $y = -1$ . So problem arises when  $y = -1$ .

(c) Solve the IVP given by this ODE and the condition  $y(1) = 1$ .

**Solution**

$$\frac{dy}{dx} = \frac{x}{1+y}$$

is a separable equation and formally can be rewritten as:

$$(y+1)dy = xdx$$

Integrate both sides to get:

$$y^2/2 + y = x^2/2 + C$$

$$y^2 + 2y = x^2 + C$$

$$(y+1)^2 = x^2 + C$$

Plug in the initial value  $y(1) = 1$

$$(1+1)^2 = 1 + C \rightarrow C = 3$$

$$(y + 1)^2 = x^2 + 3$$

$$y = -1 + -\sqrt{x^2 + 3}$$

The  $+ -$  sign comes from the fact that  $\sqrt{x^2} = + - x$  but only the  $+$  satisfies the initial condition  $y(1) = 1$ . So we have:

$$y = -1 + \sqrt{x^2 + 3}$$