AM 33, Fall 2001, Third Practice Exam (Solution)

1. Let

$$f(t) = \begin{cases} 2 & ; \quad 0 \le t < 2 \\ t & ; \quad t > 2 \end{cases} .$$

Compute the Laplace transform of f. Solve the IVP

$$y'(t) + 2y(t) = f(t), \qquad y(0) = 1.$$

Solution: Note that

$$f(t) = 2 + u_2(t) \cdot (t-2); \quad t \ge 0.$$

Hence

$$F(s) = \mathcal{L}\{f\} = \mathcal{L}\{2\} + e^{-2s}\mathcal{L}\{t\} = \frac{2}{s} + e^{-2s}\frac{1}{s^2}.$$

Taking Laplace transforms on both sides of the differential equation, we have

$$\big[sY - y(0)\big] + 2Y = (s+2)Y - 1 = F(s) = \frac{2}{s} + e^{-2s}\frac{1}{s^2},$$

or

$$Y(s) = \frac{1}{s} + e^{-2s}\frac{1}{s^2(s+2)}.$$

With partial fraction expansion, we have

$$\frac{1}{s^2(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2}$$

for some sonstant A, B, C. It is not difficult to determine (check!)

$$A = -\frac{1}{4}, \quad B = \frac{1}{2}, \quad C = \frac{1}{4}.$$

Therefore,

$$h(t) \doteq \mathcal{L}^{-1}\left(\frac{1}{s^2(s+2)}\right) = -\frac{1}{4} + \frac{t}{2} + \frac{1}{4}e^{-2t} = \frac{t}{2} + \frac{1}{4}\left(e^{-2t} - 1\right).$$

It follows that

$$y(t) = \mathcal{L}^{-1}\{Y\} = 1 + u_2(t)h(t-2).$$

2. Solve the following integro-differential equation by taking Laplace transform

$$y'(t) = 2y(t) + 2e^{3t} + 2\int_0^t e^{3(t-\tau)} y(\tau)d\tau, \qquad y(0) = 1.$$

Solution: Rewrite the equation as

$$y' = 2y + 2e^{3t} + 2(e^{3t} * y).$$

Let $Y = \mathcal{L}\{y\}$, and thus $\mathcal{L}\{y'\} = sY - y(0) = sY - 1$. Taking Laplace transform on both sides of the equation, we have

$$sY - 1 = 2Y + \frac{2}{s-3} + 2\frac{1}{s-3} \cdot Y$$

or

$$\left(s - 2 - \frac{2}{s-3}\right) Y = \frac{s-1}{s-3} \qquad \Rightarrow \qquad Y = \frac{s-1}{s^2 \quad 5s + 4} = \frac{1}{s-4}.$$

It follows that

$$y(t) = \mathcal{L}^{-1}\{Y\} = e^{4t}.$$

3. Using Laplace transform method to solve the following system of equations:

$$\begin{aligned} \frac{dz_1}{dt} + \frac{dz_2}{dt} &= 1 - 4z_1 \\ \frac{dz_1}{dt} &= 2z_1 - z_2 + t^2 \end{aligned}$$

with initial condition $z_1(0) = 2, z_2(0) = -1$. Note: the computation here is a bit messy.

Solution: Let $\mathcal{L}(z_1) = Z_1$ and $\mathcal{L}(z_2) = Z_2$. It follows that

$$\mathcal{L}(z_1') = sZ_1 - z_1(0) = sZ_1 - 2, \qquad \mathcal{L}(z_2') = sZ_2 - z_2(0) = sZ_2 + 1,$$

Taking Laplace transform on both sides of the two equations, we obtain

$$\begin{aligned} (sZ_1 - 2) + (sZ_2 + 1) &= \frac{1}{s} - 4Z_1 \\ sZ_1 - 2 &= 2Z_1 - Z_2 + \frac{2}{s^3} \end{aligned}$$

or

$$\begin{aligned} (s+4)Z_1 + sZ_2 &= \frac{1}{s} + 1 \\ (s-2)Z_1 + Z_2 &= \frac{2}{s^3} + 2 \end{aligned}$$

Multiplying the second equation by $-s$ and adding to the first, we have

$$(3s + 4 - s^2)Z_1 = \frac{1}{s} + 1 - \frac{2}{s^2} - 2s \quad \Rightarrow \quad Z_1 = \frac{2s^3 - s^2 - s + 2}{s^2(s^2 - 3s - 4)}$$

However,

$$s^2(s^2 - 3s - 4) = s^2(s-4)(s+1), \quad 2s^3 - s^2 - s + 2 = (s+1)(2s^2 - 3s + 2).$$

It follows that

$$Z_1 = \frac{2s^2 - 3s + 2}{s^2(s-4)} = \frac{5}{8s} - \frac{1}{2s^2} + \frac{11}{8(s-4)}$$

Therefore,

$$z_1 = \frac{5}{8} - \frac{t}{2} + \frac{11}{8}e^{4t}.$$

But from the second equation, we have

$$z_2 = 2z_1 + t^2 - z_1' \quad \Rightarrow \quad z_2 = t^2 - t + \frac{7}{4} - \frac{11}{4}e^{4t}$$

4. Consider the differential equation

$$ty'' + y' + ty = 0; \qquad t \geq 0,$$

with initial condition $y(0) = 1$, $y'(0) = 0$.

(a) Let $Y = \mathcal{L}\{y\}$. Argue that Y satisfies the differential equation

$$(s^2+1)\frac{dY}{ds} + sY = 0.$$

(b) Solve the above equation to conclude that

$$Y(s) = \frac{C}{\sqrt{1+s^2}}$$

for some constant C.

Solution: Let $Y = \mathcal{L}\{y\}$. We have

$$\mathcal{L}\{y'\} = sY - y(0) = sY - 1, \qquad \mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0) = s^2Y - s.$$

Furthermore,

$$\mathcal{L}\{ty\} = -\frac{dY}{ds}, \quad \mathcal{L}\{ty''\} = -\frac{d}{ds}\left(s^2Y - s\right) = -2sY - s^2\frac{dY}{ds} + 1.$$

It follows that

$$0 = \mathcal{L}\{ty'' + y' + ty\} = \left(-2sY - s^2\frac{dY}{ds} + 1\right) + (sY - 1) - \frac{dY}{ds},$$

or equivalently,

$$(s^2+1)\frac{dY}{ds} + sY = 0.$$

This proves (a). As for (b), note that Y satisfies a linear first order ODE, which is very easy to solve. Indeed,

$$\frac{1}{Y}\frac{dY}{ds} = -\frac{s}{s^2+1} \qquad \Rightarrow \qquad \log(Y) = -\int \frac{s}{s^2+1}\,ds + c = -\frac{1}{2}\log(s^2+1) + c,$$

for some constant c, or

$$Y(s) = e^c \cdot (s^2+1)^{-\frac{1}{2}} = \frac{C}{\sqrt{s^2+1}}$$

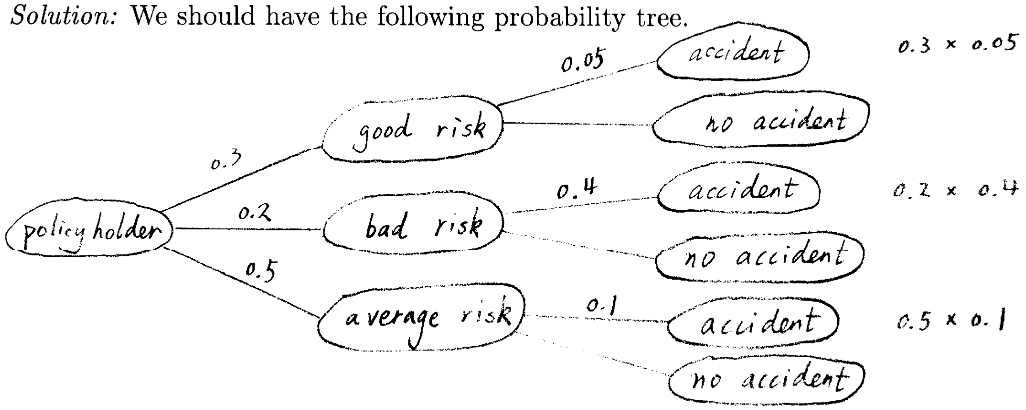
where $C = e^c$ is a constant. This proves (b).

5. An auto insurance company classifies its policyholders as "good risks", "bad risks" or "average risks": 30% are deemed "good risks", 20% are deemed "bad risks", and 50% are deemed "average risks". Historical data suggest that 5% of the "good risks", 40% of the "bad risks", and 10% of the "average risks" will be involved in an accident in the coming year.

   (a) What is the probability that a randomly chosen policyholder will involve in an accident in the coming year?

   (b) An accident claim has just been filed with the company. What is the probability that this customer was classified as a "good risk"? A "bad risk"? An "average risk"?

Solution: We should have the following probability tree.



Part (a). The probability that a randomly chosen policyholder will involve in an accident in the coming year is

$$\mathbb{P}(\text{involve in an accident}) = 0.3 \cdot 0.05 + 0.2 \cdot 0.4 + 0.5 \cdot 0.1 = 0.145.$$

Part(b). The probability that this policyholder was classified as a "good risk" is the conditional probability

$$\mathbb{P}(\text{the policyholder is a "good risk"} \mid \text{involve in an accident}).$$

By definition, this conditional probability is

$$\frac{\mathbb{P}(\text{the policyholder is a "good risk" and involves in an accident})}{\mathbb{P}(\text{involve in an accident})} = \frac{0.3 \cdot 0.05}{0.145} = \frac{3}{29}.$$

Similarly, the probabilities that this policyholder was classified as a "bad risk" or a "average risk" are $\frac{16}{29}$ and $\frac{10}{29}$, respectively. Note that these three conditional probabilities add up to 1.