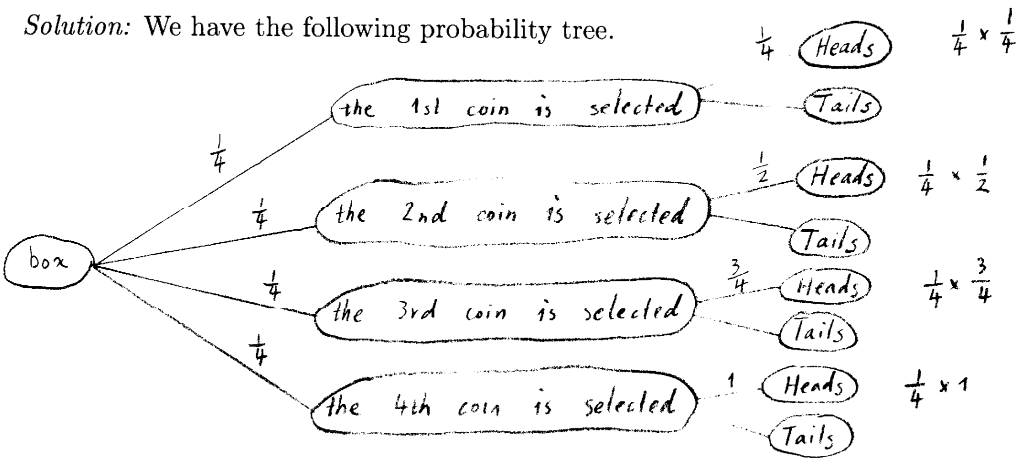
An extra probability problem

1. Suppose that a box contains four coins, for each of which there is a different probability that a Heads will be obtained when the coin is tossed. Let p_i denote the probability of a Heads when the i-th coin is tossed ($i = 1, 2, 3, 4$), and suppose that

$$p_1 = \frac{1}{4}, \ \ p_2 = \frac{1}{2}, \ \ p_3 = \frac{3}{4}, \ \ p_5 = 1.$$

Suppose that one coin is randomly selected from the box and when this coin is tossed once, a Heads is obtained. What is the probability that the i-th coin was selected?

Solution: We have the following probability tree.



For example, let us compute the probability that the first coin is selected. This probability is

$$\mathbb{P}(\text{the first coin is selected} \mid \text{the randomly selected coin is tossed and a Heads is obtained})$$

By definition, this conditional probability is the fraction

$$\frac{\mathbb{P}(\text{the first coin is selected } \textit{and} \text{ the randomly selected coin is tossed and a Heads is obtained})}{\mathbb{P}(\text{the randomly selected coin is tossed and a Heads is obtained})}$$

It is easy to see from the probability tree that the denominator is

$$\frac{1}{4} \cdot p_1 + \frac{1}{4} \cdot p_2 + \frac{1}{4} \cdot p_3 + \frac{1}{4} \cdot p_4 = \frac{1}{4} \cdot \left(\frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1 \right) = \frac{5}{8}.$$

The numerator is

$$\frac{1}{4} \cdot p_1 = \frac{1}{16}.$$

Therefore, the probability that the first coin is selected is

$$\frac{\frac{1}{16}}{\frac{5}{8}} = \frac{1}{10}.$$

Similaryly, the probability that the second coin is selected is

$$\frac{\frac{1}{4}\cdot\frac{1}{2}}{\frac{5}{8}}=\frac{1}{5};$$

the probability that the third coin is selected is

$$\frac{\frac{1}{4}\cdot\frac{3}{4}}{\frac{5}{8}}=\frac{3}{10};$$

and the probability that the fourth coin is selected is

$$\frac{\frac{1}{4}\cdot 1}{\frac{5}{8}}=\frac{2}{5}.$$