Applied Math 33

First Exam, 9 October, 2001.

Points are as indicated. Please identify in a clear manner work that you want considered for credit. You may use your text book and one (1) calculus book for reference. Calculators that are capable of solving the calculus problems are not allowed.

1. (25 pts.)

   (a) (15 pts.) Test the following differential equation for exactness. If it is exact, find the general solution and also the solution to the initial value problem with $y(1) = 2$. (You may leave solution in implicit form if need be.)

   $$(2xy - 3x^2)dx + (x^2 + y^2)dy = 0$$

   (b) (10 pts.) Find an integrating factor that makes the following differential equation exact. (Do not solve the differential equation.)

   $$(x^2 + 2y^2)dx - xydy = 0.$$

Solution 1:

(a)

$$\frac{\partial}{\partial y}(2xy - 3x^2) = 2x \text{ and } \frac{\partial}{\partial x}(x^2 + y^2) = 2x,$$

so the equation is exact. We find $\psi(x, y)$.

$$\psi_x(x, y) = 2xy - 3x^2$$

$$\Rightarrow \psi(x, y) = x^2y - x^3 + c_1(y)$$

$$\Rightarrow \psi_y(x, y) = x^2 + c_1'(y) = x^2 + y^2,$$

so $c_1(y) = \frac{1}{3}y^3$, and thus

$$x^2y(x) - x^3 + \frac{1}{3}y(x)^3 = c.$$

If $y(1) = 2$, then $2 - 1 + \frac{1}{3}8 = c$, so for the solution to the IVP $c = \frac{11}{3}$.

(b) Exactness, after multiplication by $\mu(x, y)$, requires

$$\mu_y(x, y)(x^2 + 2y^2) + \mu(x, y)(4y) = \mu_x(x, y)(-xy) + \mu(x, y)(-y).$$

If μ is a function of x only then $\mu_y = 0$, and canceling off y gives

$$4\mu(x) = x\mu_x(x) - \mu(x)$$

$$\Rightarrow \frac{d\mu}{\mu} = \frac{-5dx}{x}$$

$$\Rightarrow \ln \mu(x) = -5 \ln x$$

$$\Rightarrow \mu(x) = x^{-5}.$$

2. (25 pts.) Consider the equation

$$y' = \frac{x^2}{y(1+x^3)}.$$

(a) (10 pts.) Find a parameterized family of solutions.

(b) (15 pts.) Identify the regions in the xy-plane where the conditions of the existence and uniqueness theorem are not satisfied, and explain why.

Solution 2:

(a) The equation is separable. Thus

$$ydy = \frac{x^2 dx}{(1+x^3)}$$

$$\Rightarrow \frac{1}{2}y^2(x) = \frac{1}{3}\ln|1+x^3| + c$$

$$\Rightarrow y(x) = \pm\sqrt{\frac{2}{3}\ln|1+x^3| + c}$$

(b) A unique solution exists for data $y(x_0) = y_0$ if $f(x,y) = x^2/y(1+x^3)$ and $f_y(x,y) = -x^2/y^2(1+x^3)$ are continuous in a rectangle that contains (x_0, y_0). This is *not* true if $x_0 = -1$ or $y_0 = 0$.

3. (25 pts.) Consider the differential equation

$$y'(x) = 3y(x) - y(x)^2 - 2$$

(a) (5 pts.) Determine the critical points of this ODE.

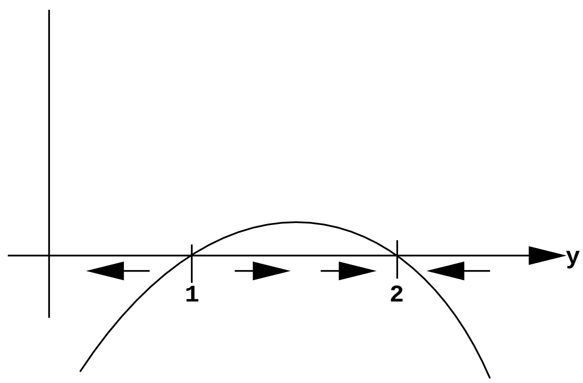
(b) (10 pts.) Determine the stability of each of these points.

(c) (10 pts.) Without solving the equation, identify the limit as x tends to ∞ of the solution that satisfies $y(0) = 3/2$. What about the solution that satisfies $y(0) = 1$?

Solution 3:

(a)

$$\begin{aligned} &3y^* - (y^*)^2 - 2 = 0 \\ &\Rightarrow (y^* - 2)(y^* - 1) = 0 \\ &\Rightarrow y^* = 1, 2. \end{aligned}$$



(b) With $f(y) = 3y - y^2 - 2$, $f'(1) > 0$, and $f'(2) < 0$. Thus 1 is unstable, 2 is stable.

(c) If $y(0) = 3/2, y(x) \to 2$ as $x \to \infty$.
If $y(0) = 1$, then $y(x) = 1$ for all x, and so $y(x) \to 1$ as $x \to \infty$.

4. (25 pts.) Consider a cylindrical tank of constant cross section A. Water is pumped into the tank at a constant rate k and leaks out through a small hole in the bottom of the tank of area a. Let $h(t)$ denote the depth of the water in the tank. It follows from Torricelli's theorem that h satisfies the differential equation

$$h'(t) = \left(k - \alpha a\sqrt{2gh(t)}\right)/A,$$

where α is a number between $1/2$ and 1 and g is the acceleration due to gravity.

(a) (15 pts.) Is there an equilibrium depth for the water in this tank with the constant input rate? If so, identify. Is this equilibrium point stable or unstable?

(b) (10 pts.) Now suppose that the water is shut off ($k = 0$), and the tank is allowed to drain. If the water is shut off at time 0 and at that time $h(0) = h_0 \geq 0$, compute the time τ when the tank empties.

Solution 4:

(a) Let

$$\theta = \alpha a\sqrt{2g}, f(h) = (k - \theta\sqrt{h})/A.$$

Then

$$f(h^*) = 0 \Rightarrow h^* = k^2/\theta^2.$$

Since

$$f^{'}(h^*) = -\theta/2\sqrt{h^*} = -1/2k < 0,$$

it is stable.

(b) Now $h^{'} = -\theta\sqrt{h}/A$, $h(0) = h_0$. The equation is separable. Thus

$$\frac{dh}{h^{1/2}} = \frac{-\theta dt}{A} \Rightarrow 2h^{1/2}(t) = \frac{-\theta t}{A} + c.$$

$$h(0) = h_0 \Rightarrow c = 2h_0^{1/2}, \text{ and so } h(\tau) = 0 \Rightarrow \frac{-\theta}{A}\tau + 2h_0^{1/2} = 0 \Rightarrow \tau = \frac{2Ah_0^{1/2}}{\theta}.$$