APPLIED MATH 33: Practice Exam 2 (Solution)

1. Find the general solution to the differential equation

$$y'' + y' - 6y = e^{2x} - 2.$$

Solution: To find a particular solution, we will split the equation into two.

$$\begin{aligned} &(1) \qquad & y'' + y' - 6y &= e^{2x} \\ &(2) \qquad & y'' + y' - 6y &= -2. \end{aligned}$$

The method we are going to use is the method of undetermined coefficients. Guess a particular solution to equation (1) of form

$$y = Ae^{2x} \quad \Rightarrow \quad y' = 2Ae^{2x}, \;\; y'' = 4Ae^{2x}$$

Substituting into (1) we have

$$y'' + y' - 6y = e^{2x}(4A + 2A - 6A) \equiv 0.$$

Try

$$y = Axe^{2x} \quad \Rightarrow \quad y' = (A + 2Ax)e^{2x}, \; y'' = (4A + 4Ax)e^{2x}$$

Substituting into (1) we have

$$e^{2x}(4A + 4Ax + A + 2Ax - 6Ax) = 5Ae^{2x} = e^{2x} \quad \Rightarrow \quad A = \frac{1}{5}$$

Therefore, a particular solution to equation (1) is $y = \frac{1}{5}xe^{2x}$.
For equation (2), guess a particular solution is a constant $y \equiv B$. Hence $y' = 0, y'' = 0$, and

$$y'' + y' - 6y = -6B = -2 \quad \Rightarrow \quad B = \frac{1}{3}$$

Therefore, a particular solution to equation (2) is $y \equiv \frac{1}{3}$.
A particluar to the original non-homogenous equation is therefore

$$y_p = \frac{1}{5}xe^{2x} + \frac{1}{3}.$$

The corresponding non-homogenous equation is

$$y'' + y' - 6y = 0 \quad \Rightarrow \quad \text{characteristic equation } \; r^2 + r - 6 = 0 \quad \Rightarrow \quad r_1 = -3, \; r_2 = 2.$$

A set of fundamental solutions is $y_1 = e^{-3x}$, $y_2 = e^{2x}$. The general solution is therefore

$$y = c_1y_1 + c_2y_2 + y_p = c_1e^{-3x} + c_2e^{2x} + \frac{1}{5}xe^{2x} + \frac{1}{3}.$$

2. One solution of

$$x^2y'' - 2xy' + (2 - 9x^2)y = 0$$

is $y_1(x) = xe^{3x}$.

- Find a second linearly independent solution y_2.
- Calculate the Wronskian $W(y_1, y_2)(x)$.
- Solve the initial value problem for the given ODE and data $y(1) = 1, y'(1) = 0$.

Solution: We let

$$y_2 = uy_1 = uxe^{3x},$$

so that

$$y_2' = u'xe^{3x} + u\left[e^{3x} + 3xe^{3x}\right],$$

$$y_2'' = u''xe^{3x} + 2u'\left[e^{3x} + 3xe^{3x}\right] + u\left[6e^{3x} + 9xe^{3x}\right].$$

Plugging in to the equation gives

$$\begin{aligned} & u''x^3e^{3x} + 2u'\left[x^2e^{3x} + 3x^3e^{3x}\right] + u\left[6x^2e^{3x} + 9x^3e^{3x}\right] \\ & \quad -2u'x^2e^{3x} - 2u\left[xe^{3x} + 3x^2e^{3x}\right] \\ & \quad\quad +(2 - 9x^2)uxe^{3x} \\ & = u''x^3e^{3x} + 6u'x^3e^{3x} = 0. \end{aligned}$$

Therefore

$$u'' + 6u' = 0.$$

Letting $v = u'$ and solving $v' + 6v = 0$ gives $v(x) = e^{-6x}$, and thus $u(x) = -\frac{1}{6}e^{-6x}$. Since the constant in front does not matter, we can take $y_2 = e^{-6x}xe^{3x} = xe^{-3x}$. The Wronskian is then

$$y_1y_2' - y_2y_1' = xe^{3x}\left[e^{-3x} - 3xe^{-3x}\right] - xe^{-3x}\left[e^{3x} + 3xe^{3x}\right] = -6x^2.$$

With

$$\begin{aligned} y(x) &= c_1xe^{3x} + c_2xe^{-3x} \\ y'(x) &= c_1\left[e^{3x} + 3xe^{3x}\right] + c_2\left[e^{-3x} - 3xe^{-3x}\right] \end{aligned}$$

the initial data imply

$$c_1e^3 + c_2e^{-3} = 1, \qquad 4c_1e^3 - 2c_2e^{-3} = 0,$$

and Gaussian elimination gives $c_1 = \frac{1}{3}e^{-3}, c_2 = \frac{2}{3}e^3$.

3. Consider the second order linear equation

$$y'' + p(t)y' + q(t)y = 0, \qquad t \geq 0,$$

where $p(t)$ and $q(t)$ are continuous functions. Suppose that $y_1(t)$ satisfies the equation together with the data $y_1(1) = 1, y_1'(1) = 3$, and that $y_2(t)$ satisfies the equation together with the data $y_2(1) = 2, y_2'(1) = 6$. Are the function y_1 and y_2 linearly independent? Why? Let $\bar{y}_1 = y_1 + y_2$ and $\bar{y}_2 = y_1 - y_2$. Are $\bar{y}_1$ and $\bar{y}_2$ linearly independent? Why?

Solution: The functions y_1 and y_2 are linearly dependent. The reason is as follows: the value of the Wronskian $W(y_1, y_2)$ at $t = 1$ equals

$$W(y_1, y_2)(1) = y_1(1)y_2'(1) - y_1'(1)y_2(1) = 1 \cdot 6 - 3 \cdot 2 = 0.$$

Since y_1 and y_2 are both solutions to the differential equation, the Wronskian is zero everywhere and the two functions are linearly dependent.

First of all, $\bar{y}_1$ and $\bar{y}_2$ are also two solutions to the differential equation. Their Wronskian at $t = 1$ is

$$W(\bar{y}_1, \bar{y}_2)(1) = \bar{y}(1)\bar{y}_2'(1) - \bar{y}_1'(1)\bar{y}_2(1) = (1+2)\cdot(3-6) - (3+6)\cdot(1-2) = -9 - (-9) = 0$$

By the same token, $\bar{y}_1$ and $\bar{y}_2$ are linearly dependent too.

4. The homogeneous equation

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0$$

has two solutions $y_1(t) = \sqrt{t}$, $y_2(t) = \frac{1}{t}$. Use the method of variation of parameters to find the general solution of the non-homogeneous equation

$$2t^2y'' + 3ty' - y = \frac{1}{t}, \qquad t > 0.$$

Solution: Consider a particular solution of form

$$y_p = u_1y_1 + u_2y_2$$

where u_1, u_2 are two functions yet to be determined. It follows that

$$y_p' = (u_1y_1' + u_2y_2') + (u_1'y_1 + u_2'y_2).$$

We obtain the first equation by letting

$$(1) \qquad u_1'y_1 + u_2'y_2 = 0, \qquad \text{or} \qquad u_1' \cdot \sqrt{t} + u_2' \cdot \frac{1}{t} = 0.$$

Therefore, we have

$$y_p'' = (u_1y_1' + u_2y_2')' = u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2'$$

and

$$\begin{aligned} 2t^2y'' + 3ty' - y &= 2t^2(u_1'y_1' + u_2'y_2') + u_1(2t^2y_1'' + 3ty_1' - y_1) + u_2(2t^2y_2'' + 3ty_2' - y_2) \\ &= 2t^2(u_1'y_1' + u_2'y_2'), \end{aligned}$$

which yields the second equation

$$(2) \qquad \frac{1}{t} = 2t^2(u_1'y_1' + u_2'y_2') = 2t^2\left(u_1' \cdot \frac{1}{2\sqrt{t}} - u_2' \cdot \frac{1}{t^2}\right).$$

Solve equations (1) and (2) to obtain

$$u_1' = \frac{1}{3}t^{-\frac{5}{2}}, \qquad u_2' = -\frac{1}{3}\frac{1}{t}.$$

In other words,

$$u_1 = -\frac{2}{9}t^{-\frac{3}{2}}, \qquad u_2 = -\frac{1}{3}\log t$$

and

$$y_p = u_1y_1 + u_2y_2 = -\frac{2}{9} \cdot \frac{1}{t} - \frac{1}{3}\frac{\log t}{t}.$$

The general solution can be written as

$$y = y_p + c_1y_1 + c_2y_2 = -\frac{1}{3}\frac{\log t}{t} + c_1\sqrt{t} + \left(c_2 - \frac{2}{9}\right)\frac{1}{t},$$

or simply as

$$y = -\frac{1}{3}\frac{\log t}{t} + c_1\sqrt{t} + c_2\frac{1}{t}$$

Here c_1, c_2 are arbitrary generic constants.