AM 33

Problem Set Number 9

Section 6.2 Problem 21 :

We want to solve the I.V.P.

$$y'' - 2y' + 2y = \cos(t) \quad ; \quad y(0) = 1 \, , \; y'(0) = 0 \qquad (1)$$

via the Laplace Transform. To do this we take the LT of both sides, ie

$$\mathcal{L}(y'' - 2y' + 2y) = \mathcal{L}(\cos t)$$

$$\Rightarrow \quad \mathcal{L}(y'') - 2\mathcal{L}(y') + 2\mathcal{L}(y) = \mathcal{L}(\cos t)$$

$$\Rightarrow \quad s^2 Y(s) - s y(0) - y'(0) - 2[sY(s) - y(0)] + 2Y(s) = \frac{s}{s^2+1}$$

$$\Rightarrow \quad s^2 Y(s) - s - 2sY(s) + 2 + 2Y(s) = \frac{s}{s^2+1}$$

$$\Rightarrow \quad Y(s)[s^2 - 2s + 2] = \frac{s}{s^2+1} + s - 2$$

$$\Rightarrow \quad Y(s) = \frac{s^3 - 2s^2 - 2s - 2}{(s^2-2s+2)(s^2+1)} .$$

$$= \frac{(s^2+1)(s-2) + s}{(s^2-2s+2)(s^2+1)}$$

$$= \frac{s-2}{s^2-2s+2} + \frac{s}{(s^2-2s+2)(s^2+1)} \qquad (2)$$

We use the partial fraction expansion to resolve the second term in equation (2). Hence we set

$$\frac{as+b}{s^2-2s+2} + \frac{cs+d}{(s^2+1)} = \frac{s}{(s^2+1)(s^2-2s+2)}$$

Expanding and equating powers of "s" yields

$$\left.\begin{array}{ll} \text{i)} & a=-c \\ \text{ii)} & b-2c+d=0 \\ \text{iii)} & a+2c-2d=1 \\ \text{iv)} & b=-2d. \end{array}\right\} \qquad \begin{array}{l} a=-\frac{1}{5} \\ b=\frac{4}{5} \\ c=\frac{1}{5} \\ d=-\frac{2}{5} \end{array}$$

Thus Eq (2) becomes

$$Y(s) = \frac{s-2}{s^2-2s+2} + \frac{-\frac{1}{5}s+\frac{4}{5}}{(s^2-2s+2)} + \frac{\frac{1}{5}(s-2)}{(s^2+1)}$$

$$= \frac{(s-1)-1}{(s-1)^2+1} - \frac{\frac{1}{5}(s-1)}{(s^2-1)^2+1} + \frac{\frac{3}{5}}{(s^2-1)^2+1} + \frac{\frac{1}{5}(s-2)}{(s^2+1)}$$

$$= \frac{(s-1)}{(s-1)^2+1} - \frac{1}{(s-1)^2+1} - \frac{1}{5}\frac{(s-1)}{(s^2-1)^2+1} + \frac{3}{5}\frac{1}{(s^2-1)^2+1} + \frac{\frac{1}{5}s}{(s^2+1)} - \frac{\frac{2}{5}}{(s^2+1)}$$

$$\Rightarrow \quad Y(s) = \frac{4}{5}\frac{(s-1)}{(s-1)^2+1} - \frac{2}{5}\frac{1}{(s-1)^2+1} + \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1}$$

$$\Rightarrow \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{4}{5}\frac{(s-1)}{(s-1)^2+1} - \frac{2}{5}\frac{1}{(s-1)^2+1} + \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1}\right)$$

$$\Rightarrow \quad y(t) = \frac{1}{5}\left[4e^{t}\cos t - 2e^{t}\sin t + \cos t - 2\sin t\right]$$

Section 6.2 Problem 34 :

We wish to take the Laplace Transform of $f(t) = te^{at}\cos bt$, which exists since $|f(t)| \le te^{at} \le Me^{(a+1)t}$. Denote by $Q(s)$ the Laplace transform of $q(t) = e^{at}\cos bt$ ie $Q(s) = \mathcal{L}(e^{at}\cos bt)$. Then by property 19 in Table 6.2.1 we have

$$\mathcal{L}(-te^{at}\cos bt) = Q'(s) \tag{1}$$

Furthermore by Property 10 of Table 6.2.1 we have

$$Q(s) = \frac{s-a}{(s-a)^2+b^2} \tag{2}$$

and so

$$\begin{aligned}\mathcal{L}(te^{at}\cos bt) &= -\mathcal{L}(-te^{at}\cos bt)\\ &= -\frac{d}{ds}\frac{(s-a)}{(s-a)^2+b^2}\\ &= \frac{(s-a)^2-b^2}{[(s-a)^2+b^2]^2}\end{aligned}$$

Section 6.2 Problem 36 :

a) We take the LT of $y'' - ty = 0$; $y(0) = 1$, $y'(0) = 0$.

$$\mathcal{L}(y'' - ty) = \mathcal{L}(0)$$

$$\Rightarrow \mathcal{L}(y'') + \mathcal{L}(-ty) = 0$$

$$\Rightarrow s^2 Y(s) - sy(0) - y'(0) + Y'(s) = 0$$

$$\Rightarrow s^2 Y(s) - s + Y'(s) = 0$$

$$\Rightarrow Y'(s) = s - s^2 Y(s)$$

b) We want to take LT of $(1-t^2)y'' - 2ty' + \alpha(\alpha+1)y = 0$; $y(0)=0$, $y'(0)=1$

$$\mathcal{L}\left((1-t^2)y'' - 2ty' + \alpha(\alpha+1)y\right) = \mathcal{L}(0)$$

$$\Rightarrow \mathcal{L}(y'') + \mathcal{L}(-t^2y'') + 2\mathcal{L}(-ty') + \alpha(\alpha+1)\mathcal{L}(y) = 0.$$

$$\Rightarrow s^2 Y(s) - sy(0) - y'(0) - Y''(s) + 2Y'(s) + \alpha(\alpha+1)Y(s) = 0.$$

$$\Rightarrow s^2 Y(s) - 1 - Y''(s) + 2Y'(s) + \alpha(\alpha+1)Y(s) = 0$$

$$\Rightarrow Y''(s) = 2Y'(s) + \left[\alpha(\alpha+1) + s^2\right]Y(s) - 1$$

Section 6.2, Problem 36 (b)

Answer: We want to take LT of

$$(1-t^2)y'' - 2ty' + \alpha(\alpha+1)y = 0 \quad ; \qquad y(0)=0, \quad y'(0)=1.$$

However,

$$\mathcal{L}(y') = sY - y(0) = sY$$

$$\mathcal{L}(y'') = s^2Y - sy(0) - y'(0) = s^2Y - 1.$$

Therefore,

$$\mathcal{L}(ty') = -(sY)' = -sY' - Y$$

$$\mathcal{L}(t^2y'') = (s^2Y-1)'' = s^2Y'' + 4sY' + 2Y;$$

And the Laplace transform on left-hand side is

$$(s^2Y-1) - (s^2Y'' + 4sY' + 2Y) + 2(sY'+Y) + \alpha(\alpha+1)Y = 0$$

or

$$s^2Y'' + 2sY' - [\alpha(\alpha+1) + s^2]Y = -1$$

Section 6.2 Problem 37:

Note that if $g(t) = \int_0^t f(\tau)\,d\tau$ we have $g(0) = 0$ and $g'(t) = f(t)$. Thus, since on one hand

$$\mathcal{L}(g'(t)) = s\mathcal{L}(g(t)) = sG(s) - g(0) = sG(s) \qquad (1)$$

and by the defintion of $g'(t)$ as $g'(t) = f(t)$ we have

$$\mathcal{L}(g'(t)) = \mathcal{L}(f(t)) = F(s). \qquad (2)$$

Equating (2) and (1) yields

$$\boxed{G(s) = \frac{F(s)}{s}}$$