Solution to AM 33 HW 8

1. 6.1.2.

$$f(t) = \begin{cases} t^2 & 0 \le t \le 1 \\ (t-1)^{-1} & 1 < t \le 2 \\ 1 & 2 < t \le 3 \end{cases}$$

   Solution: $f(t)$ is continous at [0,1) and (1,3]. □

2. 6.1.5. Find the Laplace transform of each of the following functions:

   (a) t

   (b) t^2

   (c) t^n, where n is a positive integer.

   Solution:

   (a) $f(t) = t$

$$\begin{aligned} \mathcal{L}f(t)(s) &= \int_0^\infty e^{-st} f(t)\mathrm{d}t \\ &= \int_0^\infty e^{-st} t\mathrm{d}t \\ &= -\frac{1}{s}\int_0^\infty t\mathrm{d}e^{-st} \\ &= -\frac{1}{s}(-\int_0^\infty e^{-st}\mathrm{d}t) \\ &= \frac{1}{s}\cdot\frac{1}{s} \\ &= \frac{1}{s^2} \end{aligned}$$

   (b) for the case of $f(t) = t^2$, we discuss the more general case in (c).

   (c) $f(t) = t^n$, denote $\mathcal{L}_n = \mathcal{L}(f(t))$, then, with out putting the computing details here,

$$\mathcal{L}_n = \frac{n}{s}\mathcal{L}_{n-1}$$

   therefore,

$$\mathcal{L}_n = \frac{n!}{s^n}\mathcal{L}_0 = \frac{n!}{s^{n+1}}$$

   □

3. 6.1.14. Assuming the necessary integration formulas extend to complex case, find the Laplace transformation of the given function; a and b are real constants.

$$f(t) = e^{at}\cos bt$$

Solution:

$$\mathcal{L}(f(t))(s) = \int_0^\infty e^{-st}e^{at}\frac{e^{ibt}+e^{-ibt}}{2}\mathrm{d}t$$

$$\begin{aligned}
\int_0^\infty e^{-st}e^{at}e^{ibt}\mathrm{d}t &= \int_0^\infty e^{(a-s)t+ibt}\mathrm{d}t \\
&= \frac{-1}{(a-s)+ib} \qquad\qquad s > a \\
&= \frac{1}{s-a-ib}
\end{aligned}$$

therefore,

$$\mathcal{L}(f(t))(s) = \mathrm{Re}\frac{1}{(s-a)-ib} = \frac{s-a}{(s-a)^2+b^2}$$

□

4. 6.1.23 determine whether the given integral converges or diverges $\int_1^\infty t^{-2}e^t\mathrm{d}t$

   Solution:

$$\lim_{x\to\infty} t^{-2}e^t = \infty$$

   , so, the integral diverges. □

5. 6.1.27. show that

$$\mathcal{L}t^{-1/2} = \frac{2}{\sqrt{s}}\int_0^\infty e^{-x^2}\mathrm{d}x, \qquad s > 0$$

$$\mathcal{L}t^{-1/2} = \sqrt{\pi/s}$$

$$\mathcal{L}t^{1/2} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

   Solution:

   Referring to 2. (6.1.5), one can see :

$$\mathcal{L}t^p = \Gamma(p+1)/s^{p+1}$$

   where,

$$\Gamma(p+1) = \int_0^\infty e^{-x}x^p\mathrm{d}x$$

   just by solely calculus, one can verify the following:

$$\begin{aligned}
\Gamma(1/2) &= \int_0^\infty e^x x^{-1/2}\mathrm{d}x \\
&= 2\int_0^\infty e^{-x^2}\mathrm{d}x \\
&= \sqrt{\pi} && (1) \\
\Gamma(p+1) &= p\Gamma(p) && (2)
\end{aligned}$$

   □

6. 6.2.2, 6.2.6, 6.2.7. Find the inverse Laplace transforms of the given functions

Solutions:

(a)

$$\begin{aligned}\mathcal{L}^{-1}\frac{4}{(s-1)^3} &= 2\mathcal{L}^{-1}\frac{2}{(s-1)^{2+1}} \\ &= 2(t^2e^t)\end{aligned}$$

(b)

$$\begin{aligned}\mathcal{L}^{-1}(\frac{2s-3}{s^2-4} &= \mathcal{L}^{-1}(\frac{2s}{s^2-4}) - 3\mathcal{L}^{-1}(\frac{1}{s^2-4}) \\ &= 2\mathcal{L}^{-1}\frac{s}{s^2-4} - \frac{3}{2}\mathcal{L}^{-1}\frac{2}{s^2-4} \\ &= 2\cosh 2t - \frac{3}{2}\sinh 2t\end{aligned}$$

$$\begin{aligned}\mathcal{L}^{-1}\frac{2s+1}{s^2-2s+2} &= \mathcal{L}^{-1}\frac{2(s-1)+3}{(s-1)^2+1} \\ &= 2\mathcal{L}^{-1}\frac{s-1}{(s-1)^2+1} + 3\mathcal{L}^{-1}\frac{1}{(s-1)^2+1} \\ &= 2e^t\cos t + 3e^t\sin t\end{aligned}$$

□

7. 6.2.19. $y^{(4)} - 4y = 0; \qquad y(0) = 1, y'(0) = 0, y''(0) = -2, y'''(0) = 0$

Solution:

take Laplace transform on both sides of the eqn, we get

$$\mathcal{L}(y^{(4)} - 4\mathcal{L}(y) = 0$$

while

$$\begin{aligned}\mathcal{L}y^{(4)} &= s^4\mathcal{L}(y) - s^3y(0) \\ &\quad -s^2y'(0) - sy''(0) - y'''(0) \\ &= s^4\mathcal{L}(y) - s^3 + 2s\end{aligned}$$

plug back to the eqn,

$$(s^4-4)\mathcal{L}(y) - s^3 + 2s = 0$$

$$\mathcal{L}(y) = \frac{s(s^2-2)}{s^4-4} = \frac{s}{s^2+2}$$

therefore

$$y = \cos\sqrt{2}t$$

□