Fall 2001, AM33 Solution to hw7

1. Section 3.4, problem 41 We are solving the ODE

$$t^2y'' + 3ty' + 1.25y = 0$$

By problem 38 $x = \log t$ turns this DE into a constant coefficient DE.

$$x = \log t \Rightarrow t = e^x \Rightarrow \frac{dt}{dx} = e^x = t$$

By the chain rule

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}t$$

We differentiate the last term and by again the chain rule:

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}\frac{dt}{dx}t + \frac{dy}{dt}\frac{dt}{dx} = \frac{d^2y}{dt^2}t^2 + \frac{dy}{dt}t$$

Thus the ODE becomes:

$$\underbrace{t^2\frac{d^2}{dt^2} + t\frac{dy}{dt}}_{\frac{d^2y}{dx^2}} + 2\underbrace{t\frac{dy}{dt}}_{\frac{dy}{dx}} + 1.25y = 0$$

The ODE

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + 1.25y$$

has the characteristic polynomial $r^2 + 2r + 1.25$ which has roots: $-1 + i/2$, $-1 - i/2$. Thus the general solution is:

$$c_1e^{-x}\sin x/2 + c_2e^{-x}\cos x/2 = c_1t^{-1}\sin(\log(t)/2) + c_2t^{-1}\cos(\log(t)/2)$$

2. Section 3.5, problem 26

We are given

$$t^2y'' - t(t+2)y' + (t+2)y = 0$$

and the solution $y_1(t) = t$. We assume the other solution is of the form $y(t) = tu(t)$. Then $y'(t) = u(t) + u'(t)t$, $y''(t) = u'(t) + u''(t)t + u'(t)$. We substitute these in the DE:

$$t^2(u'(t) + u''(t)t + u'(t)) - t(t+2)(u(t) + u'(t)t) + (t+2)(tu(t)) = 0$$

$$t^3(u''(t) - u'(t)) = 0 \Rightarrow u''(t) = u'(t) \Rightarrow u'(t) = ce^t \Rightarrow u(t) = ce^t \Rightarrow y(t) = cte^t$$

3. Section 3.5, problem 32

We are given the DE

$$y'' + \delta(xy' + y) = 0$$

We first verify that $y_1(x) = e^{-\delta x^2/2}$ is a solution.

$$y_1'(x) = -\delta x e^{-\delta x^2/2}$$

$$y_1''(x) = -\delta e^{-\delta x^2/2} + \delta^2 x^2 e^{-\delta x^2/2}$$

$$y_1''(x) + \delta(xy_1'(x) + y_1(x)) = -\delta e^{-\delta x^2/2} + \delta^2 x^2 e^{-\delta x^2/2} + \delta x(-\delta x e^{-\delta x^2/2}) + \delta e^{-\delta x^2/2} = 0$$

So indeed $y_1(x)$ is a solution. To reduce the order we set $y(x) = y_1(x)u(x)$.

$$y' = y_1'u + y_1u' = -\delta x e^{-\delta x^2/2}u + e^{-\delta x^2/2}u'$$

$$y'' = y_1''u + y_1'u' + y_1'u' + y_1u'' = (-\delta e^{-\delta x^2/2} + \delta^2 x^2 e^{-\delta x^2/2})u + 2(-\delta x e^{-\delta x^2/2})u' + e^{-\delta x^2/2}u''$$

Substitute these in the original DE and get the following DE:

$$e^{-\delta x^2/2}(u'' - \delta x u') = 0 \Rightarrow u'' - \delta x u' = 0 \Rightarrow u' = Ce^{\delta x^2/2} \Rightarrow u(x) = C\int_0^x e^{\delta t^2/2}dt$$

Then the other solution is:

$$y(x) = y_1(x)u(x) = Ce^{-\delta x^2/2}\int_0^x e^{\delta t^2/2}dt$$

4. Section 3.6, Problem 15 We want to solve the following IVP:

$$y'' - 2y' + y = te^t + 4, \qquad y(0) = 1, y'(0) = 1$$

We observe that the non-homogeneous part of the equation $te^t + 4$ is a solution to the homogeneous part. Thus the particular solution should be of the form (page, 177) $t^2e^t(At+B)+C$ When we substitute this is in the above DE we see $A = 1/6, B = 0, C = 4$ The homogeneous equation has the solution $C_1te^t + C_2e^t$. Thus the general solution to the homogeneous problem is:

$$C_1te^t + C_2e^t + \frac{1}{6}t^3e^t + 4$$

Using the initial conditions we find $C_1 = 4, C_2 = -3$

5. Section 3.6, Problem 17

The DE we are given is:

$$y'' + 4y = 3\sin 2t$$

Once again the righthand side of this equation is a solution to the homogeneous equation. Thus the particular solution should be of the form (page 177) $t(A\cos 2t + B\sin 2t)$. Once we substitute this in the above equation we find $A = -3/4, B = 0$. The general solution to the homogeneous equation is $C_1\sin 2t + C_2\cos 2t$. Thus the general solution of the DE is $C_1\sin 2t + C_2\cos 2t - 3/4t\cos 2t$. Using the initial conditions one finds $C_1 = -1/8, C_2 = 2$.

6. Section 3.7, problem 15

$y_1(t) = 1 + t, y_2(t) = e^t$.

$$y_1'(t) = 1, y_1''(t) = 0$$

$$y_2'(t) = e^t, y_2''(t) = e^t$$

Corresponding homogeneous equation is:

$$y'' - \frac{(1+t)}{t}y' + \frac{1}{t}y = 0$$

(Important thing to do here is to put the DE into the form in which y'' has no coefficient in front of it) Substitute y_1, y_1', y_1'' in this expression:

$$-(1+t)/t + (1+t)/t = 0$$

likewise it can be seen y_2 satisfies the homogeneous DE.

To find a particular solution we assume it has the form:

$$u_1(t)(t+1) + u_2(t)e^t$$

We further put the restriction that

$$u_1'(t)(t+1) + u_2'(t)e^t = 0$$

This entails that (page 182)

$$u_1'(t) + u_2'(t)(e^t) = te^{2t}$$

(Again note that $g(t)$ is not t^2e^{2t} but rather (after dividing by the t in front of y''): te^{2t})

This is a linear system where the unknowns are $u_1'(t), u_2'(t)$. Solving this system we get:

$$u_1'(t) = -e^{2t}, u_2'(t) = e^t(t+1)$$

Integrating these we get

$$u_1(t) = -\frac{1}{2}e^{2t}, u_2(t) = te^t$$

. Thus the particular solution is:

$$-(t+1)\frac{1}{2}e^{2t} + te^{2t} = \frac{(t-1)}{2}e^{2t}$$

7. Section 3.7, problem 21

   Let $u(t)$ be the unique solution of the initial value problem:

   $$y'' + p(t)y' + q(t)y = 0, y(t_0) = y_0, y'(t_0) = y_0'$$

   By the existence-uniqueness theorem such a u exists.

   Likewise define $v(t)$ to be the unique solution of the initial value problem:

   $$y'' + p(t)y' + q(t)y = g(t), y(t_0) = y'(t_0) = 0$$

   Again by the existence-uniqueness theorem such a v exists. So all we need observe that $u+v$ satisfies the given initial value problem:

   $$\begin{aligned}&(u+v)'' + p(t)(u+v)' + q(t)(u+v) =\\&[u'' + p(t)u' + q(t)u] + [v'' + p(t)v' + q(t)v] =\\&0 + g(t) = g(t)\end{aligned}$$

So the DE is satisfied. We also observe

$$(u+v)(t_0) = u(t_0) + v(t_0) = y_0 + 0 = y_0, (u+v)'(t_0) = u'(t_0) + v'(t_0) = y_0' + 0 = y_0'$$

Hence $u+v$ solves the given initial value problem. By the uniqueness theorem it is the only solution.

8. Section 3.7, problem 22

$$Y(t) = \int_{t_0}^{t} \frac{y_1(s)y_2(t) - y_1(t)y_2(s)}{y_1(s)y_2'(s) - y_1'(s)y_2(s)} g(s)ds$$

By the discussion on page 183, we know that $Y(t)$ satisfies the DE

$$y'' + p(t)y' + q(t) = g(t)$$

The only thing to check are the initial conditions:

$$Y(t_0) = \int_{t_0}^{t_0} ... = 0$$

So the first condition is satisfied. Now remember we assumed that the solution is of the form $Y(t) = y_1(t)u_1(t) + y_2(t)u_2(t)$ and that we assumed $u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$ which entailed:

$$Y'(t) = u_1(t)y_1'(t) + u_2(t)y_2'(t)$$

The $Y(t)$ given above is:

$$Y(t) = [\int_{t_0}^{t} \frac{-y_2(s)}{y_1(s)y_2'(s) - y_1'(s)y_2(s)} g(s)ds]y_1(t) + [\int_{t_0}^{t} \frac{y_1(s)}{y_1(s)y_2'(s) - y_1'(s)y_2(s)} g(s)ds]y_2(t)$$

Thus

$$u_1(t_0) = \int_{t_0}^{t_0} .... = 0, u_2(t_0) = \int_{t_0}^{t_0} .... = 0$$

which gives:

$$Y'(t) = 0y_1'(t_0) + 0y_2'(t_0) = 0$$

So Y(t) satisfies this condition too. Hence it is the solution to the initial value problem.

9. last question

As suggested by the hint we will solve two seperate problems and then merge them to solve the given problem.

Consider the IVP

$$y'' + 4y' + 5y = 0, \qquad y(0) = 0,\ y'(0) = 1. \tag{1}$$

The characteristic polynomial is $r^2 + 4r + 5$ which has two roots

$$r_1 = -2 + i, \qquad r_2 = -2 - i.$$

Therefore,

$$y_1 = e^{-2t}\cos t, \qquad y_2 = e^{-2t}\sin t$$

and the general solution is

$$u = c_1 y_1 + c_2 y_2.$$

Using the initial condition $y(0) = 0,\ y'(0) = 1$, we can determine $c_1 = 0,\ c_2 = 1$, or we have

$$u = y_2 = e^{-2t}\sin t$$

is a solution to the IVP (1). Now consider the IVP

$$y'' + 4y' + 5y = g(t), \qquad y(0) = 0,\ y'(0) = 0. \tag{2}$$

Note the Wronskian $W(y_1, y_2) = e^{-4t}$. By the preceding problem, we have

$$v = \int_0^t e^{2(s-t)}\sin(t-s)\,g(s)\,ds$$

is a solution to (2). Therefore,

$$u + v = e^{-2t}\sin t + \int_0^t e^{2(s-t)}\sin(t-s)\,g(s)\,ds$$

is a solution to the IVP

$$y'' + 4y' + 5y = g(t), \qquad y(0) = 0,\ y'(0) = 1.$$