Solutions to Homework No. 6

SECTION 3.3

Problem 4:
Using Theorem 3.3.1, all we have to do is check whether the Wronskian of $f(x) = \exp(3x)$ and $g(x) = \exp(3(x-1))$ is identically zero or not. Since $W(f,g) := fg' - f'g = 3(x-1)\exp(3x)\exp(3(x-1)) - 3\exp(3x)\exp(3(x-1)) = 3(x-2)\exp(6x-3) \neq 0$ identically we find that f, g are linearly independent.

Problem 7:
Let $f(t) = 3t$ and $g(t) = |t|$ (since function g is not differentiable at $t = 0$, the Wronskian is not defined at $t = 0$). Note that $g(t) = t = \frac{1}{3}f(t)$ for $t \geq 0$ and $g(t) = -t = -\frac{1}{3}f(t)$ for $t \leq 0$. It is easy to see from definition that f, g are linearly independent if and only if 0 is an interior point of the interval.

Problem 12:
Let $y_3 = y_1 + y_2$ and $y_4 = y_1 - y_2$. Then $W(y_3, y_4) = y_3 y_4' + y_3' y_4 = (y_1 + y_2)(y_1' - y_2') - (y_1' + y_2')(y_1 - y_2) = -2(y_1 y_2' - y_1' y_2) = -2W(y_1, y_2)$ Since $W(y_3, y_4) = -2W(y_1, y_2)$ we see by algebraic equality that $W(y_3, y_4) \neq 0$ if and only if $W(y_1, y_2) \neq 0$. Thus y_3, y_4 are linearly independent if and only if y_1, y_2 are.

Problem 17:
First we write the Bessel equation in standard form by dividing by x^2 to get

$$y'' + \frac{1}{x}y' + \frac{(x^2 - \nu^2)}{x^2}y = 0 \qquad x \neq 0.$$

Comparing with Equation (7) of Theorem 3.3.2 we find that $p = 1/x$ and so if y_1, y_2 are two solutions of the Bessel equation then $W(y_1, y_2)(t) = C\exp\left(-\int p(x)dx\right) = C\exp(-\ln(x)) = \frac{C}{x}$.

Problem 25:
Again we use Theorem 3.3.2, which tells us that the Wronskian is always zero for $t \in I$ or it is never zero for $t \in I$. Hence to show that two solutions are linearly dependent it suffices to show that their Wronskian is zero at some point $t_0 \in I$. However, if y_1, y_2 are such that they have the same maxima or minima in I, then this means that there exists some $t_0 \in I$ such that $y_1'(t_0) = y_2'(t_0) = 0$ and so $W(y_1, y_2)(t_0) = (y_1 y_2' - y_1' y_2)(t_0) = 0$ which means y_1, y_2 are linearly independent.

Problem 27:
Since $W(t, t^2) = t^2$ which is not identically zero on $-1 < t < 1$, then by Theorem 3.3.1 t, t^2 are linearly independent. Since $W(t, t^2) = 0$ for $t = 0$ we see that t, t^2 cannot be solutions to Equation (7). However, setting $u = t$ and $v = t^2$ we have $t^2 u'' - 2tu' + 2u = -2 + 2 = 0$, and $t^2 v'' - 2tv' + 2v = 2t^2 - 4t^2 + 2t^2 = 0$ and so t, t^2 are solution to some second-order linear differential equation. This is not a contradiction at all because for Theorem 3.3.2 to hold we require continuity of the coefficients $p(t), q(t)$, and for the differential equation in consideration we have (after dividing by t^2) $p(t) = 2/t$ and $q(t) = 2/t^2$ which are certainly not

continuous on any interval containing $t = 0$.

SECTION 3.4

Problem 19:
The characteristic equation for $y'' - 2y' + 5y = 0$ is $r^2 - 2r + 5 = 0$ which has solutions $r = 1 \pm 2i$. Thus the solutions are of the form $y = c_1 e^t \cos(2t) + c_2 e^t \sin(2t)$. Imposing the condition $y(\pi/2) = 0$ yields $c_1 e^{\pi/2} \cos(\pi) + c_2 e^{\pi/2} \sin(\pi)$ which implies $c_1 = 0$. Imposing the condition $y'(\pi/2) = 2$ yields $-2c_1 e^{\pi/2} \sin(\pi) + 2c_2 e^{\pi/2} \cos(\pi) = -c_2 e^{\pi/2} = 2$ which implies $c_2 = -e^{-\pi/2}$. Therefore the solution to this problem is $y = -e^{(t-\pi/2)} \sin(2t)$.

Problem 32:
Let $\phi(t) = u(t) + iv(t)$ be a soultion to

$$y'' + p(t)y' + q(t)y = 0 \qquad (*)$$

where u, v are real. Then since

$$\begin{aligned} 0 &= \phi'' + p(t)\phi' + q(t)\phi \\ &= u'' + iv'' + p(t)(u' + iv') + q(t)(u + iv) \\ &= \big(u'' + p(t)u' + q(t)u\big) + i\big(v'' + p(t)v' + q(t)v\big) \end{aligned}$$

the only way this can hold is that if both the real part and the imaginary part are zero; i.e. if u and v both satisfy the differential equation (*) which implies that both u and v are solutions too. One the other hand, by algebraic equality we see that if u, v are both solutions to (*) then so is ϕ.

SECTION 3.5

Problem 11:
The characteristic equation for this particular ODE is $9r^2 - 12r + 4 = 0$ which has the double root $r = 2/3$. Thus the general solution of the ODE is $y = c_1 e^{2t/3} + c_2 t e^{2t/3}$. Imposing the conditions $y(0) = 2$ yields $c_1 = 2$ and imposing the condition $y'(0) = -1$ yields $(2/3)c_1 + c_2 = -1$ which implies $c_2 = -\frac{7}{3}$. Thus the solution is $y = 2e^{2t/3} - \frac{7}{3} t e^{2t/3}$.

Problem 18:
a) The characteristic equation for $9y'' + 12y' + 4y = 0$ is $9r^2 + 12r + 4 = 0$ which has the double root $r = -2/3$. Thus a family of solutions for this ODE is $y(t) = c_1 e^{-2t/3} + c_2 t e^{-2t/3}$. The initial condition $y(0) = a > 0$ yields $c_1 = a$. Since $y'(t) = -2/3 c_1 e^{-2t/3} + c_2 e^{-2t/3} - 2/3 c_2 t e^{-2t/3}$, the condition $y'(0) = -1$ yields $-2/3 c_1 + c_2 = -1$ which implies $c_2 = 2a/3 - 1$. Therefore, the solution to the IVP is $y(t) = a e^{-2t/3} + (2a/3 - 1) t e^{-2t/3}$.

b) On the interval $I = [0, +\infty)$, $te^{-2t/3}$ is always positive and in fact for large t, $te^{-2t/3}$ dominates $e^{-2t/3}$. Thus in order for the solution to stay positive for all time, the coefficient of the $te^{-2t/3}$ term must be non-negative. This means that $2a/3 - 1 \geq 0$ which implies $a \geq 3/2$.