Solution to AM 33 HW 5

1. 3.1.15. Solve $y'' + 8y' - 9y = 0, \qquad y(1) = 1, \qquad y'(1) = 0$

   Solution:

   the characteristic equation is:

$$x^2 + 8x - 9 = 0$$

$$x_{1,2} = 1, -9$$

   let

$$y = ae^{x-1} + be^{-9(x-1)}$$

   then

$$y' = ae^{x-1} - 9be^{-9(x-1)}$$

   plug the initial value in:

$$\begin{cases} a + b = 1 \\ a - 9b = 0 \end{cases}$$

   solve it, get

$$b = \frac{1}{10}, \qquad a = \frac{9}{10}$$

   so the solution to the ODE is

$$y = 9/10e^{x-1} + 1/10e^{-9(x-1)}$$

2. 3.2.5 Find Wronski for given pair $e^t \sin t, \quad e^t \cos t$

   Solution:

$$\begin{aligned} W &= \begin{vmatrix} e^t \sin t & e^t \cos t \\ e^t \cos t + e^t \sin t & e^t \cos t - e^t \sin t \end{vmatrix} \\ &= e^{2t} \sin t \cos t - e^{2t} \sin^2 t - e^{2t} \cos^2 t - e^{2t} \sin t \cos t \\ &= -e^{2t} \qquad \square \end{aligned}$$

3. 3.2.13 Verify $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are two solutions of the ODE $t^2 y'' - 2y = 0$ for $t > 0$. also show $c_1 t^2 + c_2 t^{-1}$ is also a solution for any c_1, c_2

   Solution:

   do the latter part. let

$$y = c_1 t^2 + c_2 t^{-1}$$

$$y' = 2c_1 t - c_2 t^{-2}$$

$$y'' = 2c_1 + 2c_2 t^{-3}$$

   it is easy to see that $t^2 y'' - 2y = 0$, from above. $\square$

4. 3.2.14 Verify $y_1(t) = 1$ amd $y_2(t) = t^{1/2}$ are solutions of the ODE $yy'' + (y')^2 = 0, \quad$ for $t > 0$ and $c_1 + c_2 t^{1/2}$ is not, in general the solution

   Solution:

   let $y = c_1 + c_2 t^{1/2}$, then

$$y' = \frac{c_2}{2} t^{-1/2}$$
$$y'' = -\frac{c_2}{4} t^{-3/2}$$

   $y(t)$ is the solution, if and only if:

$$-(c_1 + c_2 t^{1/2}) \frac{c_2}{4} t^{-3/2} + \frac{c_2^2}{4} t^{-1} = 0$$
$$-\frac{c_1}{4} c_2 t^{-3/2} = 0 \qquad \text{for} \quad t > 0$$

   so $y(t)$ as above is the solution to the ODE, if and only if $c_1 c_2 = 0$. □

5. 3.2.15.Can $y = \sin(t^2)$ be a solution on an interval containing $t = 0$ of an equation $y'' + p(t)y' + q(t) = 0$ with continuous coefficients?

   Solution:

   No.

   otherwise, if $y = \sin(t^2)$ satisfy the ODE $y'' + p(t)y' + q(t) = 0$,

$$\begin{cases} y = \sin(t^2) \\ y' = 2t \cos t^2 \\ y'' = 2\cos t^2 - 4t^2 \sin t^2 \end{cases}$$

   plug back to the equation, and get:

$$2\cos t^2 - 4t^2 \sin t^2 + 2t \cos t^2 p(t) + q(t) \sin t^2 = 0 \qquad \text{for} \quad \forall t \in \text{a neighborhood of } 0$$

   but when $t = 0$, LHS=2, while RHS=0, contradiction! □

6. 3.2.19. if $W(f, g)$ is the Wronskian of f and g, and if $u = 2f - g$, $v = f + 2g$, find $W(u, v)$

   Solution:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

   let $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$, then

$$\begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} f \\ g \end{pmatrix}$$

$$
\begin{aligned}
W\begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} u & u' \\ v & v' \end{pmatrix} \\
&= \left(A\begin{pmatrix} f \\ g \end{pmatrix}, A\begin{pmatrix} f' \\ g' \end{pmatrix}\right) \\
&= A\begin{pmatrix} f & f' \\ g & g' \end{pmatrix} \\
&= AW(f,g) \\
&= \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} W(f,g) \\
&= 5W(f,g) \qquad \square
\end{aligned}
$$

7. 3.2.21 find the fundamental set of solutions for given ODE. $y'' + y' - 2y = 0, \qquad t_0 = 0$

   Solution:

   the characteristics equation of the ODE is: $x^2 + x - 2 = 0$, solve it, $x_{1,2} = 1, -2$ so the fundamental solution is: $e^x, \quad e^{-2x}$. $\square$

8. 3.2.25 $x^2y'' - x(x+2)y' + (x+2)y = 0, \qquad x > 0\ ; \qquad y_1(x) = x, \quad y_2(x) = xe^x$

   Solution:

   only do the verification part for $y_2(x)$,

$$
\begin{cases} y(x) = xe^x \\ y'(x) = xe^x + e^x = (x+1)e^x \\ y''(x) = xe^x + 2e^x = (x+2)e^x \end{cases}
$$

   plug back to the equation and got the result trivially.

$$
W(y_1, y_2) = \begin{pmatrix} x & xe^x \\ 1 & xe^x + e^x \end{pmatrix}
$$

   $W = x^2e^x \neq 0, \qquad \text{for} \quad x \neq 0,$

   so the given $y_1, \quad y_2$ did form a fundamental solution. $\square$

9. **Solution**

   similar as discussed in Problem 3.2.19,

$$
\left\{ \begin{array}{l} \phi_1 = ay_1 + by_2 \\ \phi_2 = cy_1 + dy_2 \end{array} \right\}
$$

   so,

$$
\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}
$$

   so, the same discussion as in Problem 3.2.19.

$$
W(\phi_1, \phi_2) = det[A]W(y_1, y_2)
$$

i.e.

$$W(\phi_1, \phi_2) = (ad - bc)W(y_1, y_2)$$

from this expression, we know, (ϕ_1, ϕ_2) form a fundamental solution set, if and only if $ad - bc \neq 0$. $\square$