Fall 2001, AM33 Solution to hw4

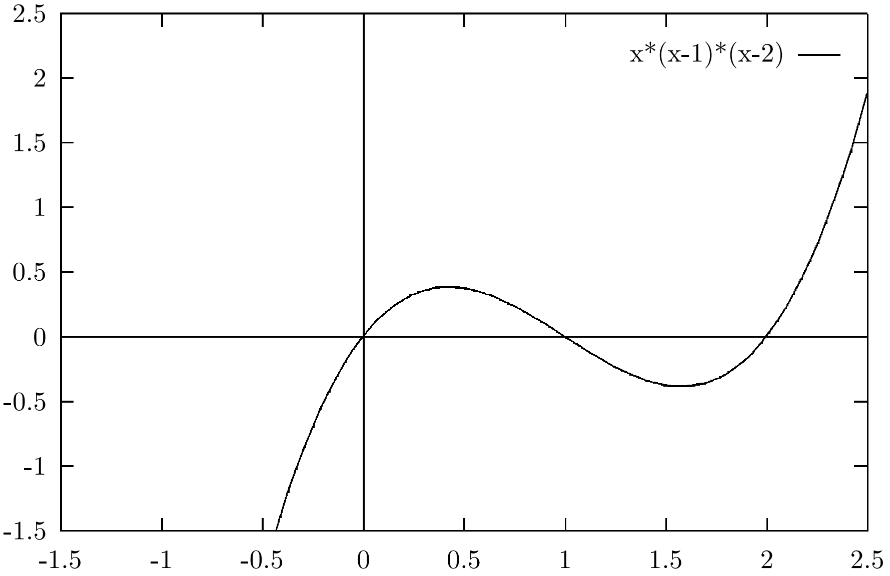
1. Problem 3, page 84

   Sketch the graph of $f(y)$ versus y, determine the critical points, and classify each one as asymptotically stable or unstable.

   $$\frac{dy}{dt} = y(y-1)(y-2)$$

   Solution

   The required graph is:

   The critical points of y are the roots of $f(y)$: 0,1,2. 0 is asymptotically unstable because for $y < 0$ $\frac{dy}{dt} = f(y) < 0$. That means y is decreasing if $y < 0$ so if at any time $y < 0$ y will further decrease and hence will never get close to 0. And similarly for $1 > y > 0$ $\frac{dy}{dt} = f(y) > 0$ which means y is increasing when $y \in (0,1)$ which will also keep y away from 0. Hence 0 is an unstable point : we stay at 0 iff we start at 0 and we never approach to 0 if we start anywhere else. Similar analysis (that is looking at the sign of the derivative (i.e. $f(y)$) in a small neighborhood of the critical point and determining whether y would get closer to the critical point or it would be repelled by it) will show that 1 is stable and 2 is unstable.

2. Problem 18, page 86 A pond forms as water collects in a connical depression of radius a and depth h. Suppose that water flows in at a constant rate k and is lost through evaporation at a rate proportional to the surface area.

   (a) Find the DE that governs the volume of the water in the pond.

   Solution

   It is useful to draw sketch of the problem:

   As can be seen from the figure that the water in the pond is always in the shape of a smaller cone, and also the smaller cones radius $r(t)$ and its height $l(t)$ are at a constant

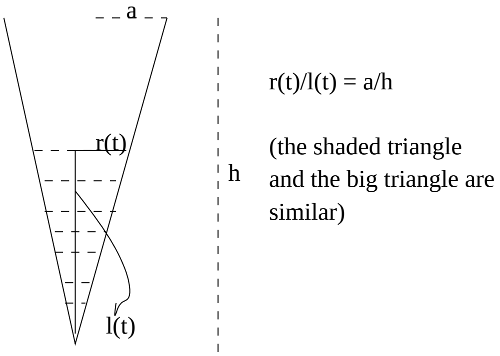


Figure 1: figure for the pond problem

proportion $\frac{a}{h}$. So we can write the height of the water in the pond at any time $l(t)$ in terms of the radius $r(t)$ of the surface of the water: $l(t) = \frac{hr(t)}{a}$. Let's call the amount of water in the pond V. Since the shape of the water is a cone, V can be written in terms of r and l:

$$V = \frac{1}{3}\pi r^2 l = \frac{1}{3a}\pi r^3 h$$

or

$$r = (\frac{3aV}{\pi h})^{\frac{1}{3}}$$

The surface area of the water is $S = \pi r^2$:

$$S = \pi(\frac{3aV}{\pi h})^{\frac{2}{3}}$$

Now we have everything in terms of V. Here is the DE that determines how V changes:

$$\frac{dV}{dt} = k - S\alpha$$

α is a constant. It is there because we only know that the evaporation is proportional to S, not equal to it. Now plug in the expression for S in terms of V to get the final result:

$$f(V) = \frac{dV}{dt} = k - \alpha\pi(\frac{3aV}{\pi h})^{\frac{2}{3}}$$

(b) Find the equilibrium depth of the water in the pond. Is the equilibrium point asymptotically stable?

Solution The equilbrium points are the roots of the last expression:

$$\frac{dV}{dt} = k - \alpha\pi(\frac{3aV}{\pi h})^{\frac{2}{3}} = 0 \iff V = (-+)\frac{(k/\alpha\pi)^{\frac{3}{2}}\pi h}{3a}$$

Since the volume can not be 0 we discard the negative equilibrium point. f(V) is positive to the left of the equilibrium point and negative to right of it. Hence it is a stable equilibrium point. In the question equilibrium depth is asked hence we need to find

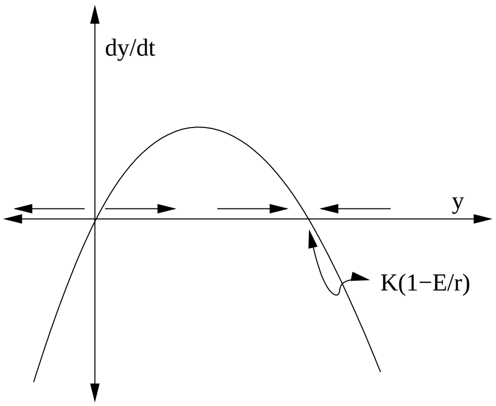


Figure 2: Figure for the fish problem

the depth $l(t)$ corresponding to this volume. To find that plug in the volume for the expression we got for the $r(t)$, that will give the radius of the surface. And depth is, as remarked earlier $\frac{r(t)h}{a}$.

(c) Find a condition that must be satisfied if the pond is not to overflow

Solution If we make sure that the pond can contain the amount of water present in the equilibrium condition then the pond would not overflow. Thus it is enough to have:

$$\frac{1}{3}\pi h a^3 = \frac{(k/\alpha\pi)^{\frac{3}{2}}\pi h}{3a}$$

or

$$k = a^{8/3}\alpha\pi$$

3. Problem 20, page 86 At a given level of effort, it is reasonable to assume that that the rate at which fish are caught depends on the population y: The more fish there are, the easier it is to catch them. Thus we assume that the rate at which fish are caught is given by Ey, where E is a positive constant with units of $1/time$, that measures the total effort made to harvest the given species of fish. To include this effect, the logistic equation is replaced by

$$\frac{dy}{dt} = f(y) = r\frac{1-y}{K}y - Ey$$

(a) Show that if $E < r$, then there are two equilibrium points, $y_1 = 0$ and $y_2 = K(1-E/r) > 0$.

Solution The equilibrium points are the roots of $f(y)$.

$$r\frac{1-y}{K}y - Ey = 0 \iff y(K - EK/r - y) = 0 \iff y = y_1 = 0 \text{ or } y = y_2 = K(1-E/r)$$

(b) Show $y = y_1$ is unstable and $y = y_2$ is unstable.

Solution As can be seen from the above figure $f(y)$ is positive to the left of y_1 (positive derivative means y increases and that is what is needed to get closer to y_1) and negative to the right of it (negative derivative meands y decreases). Hence it is a stable equilibrium point. A similar analysis shows that y_0 is unstable (look at the graph).

For the solutions of the practice exam , please refer to the web page.