Solutions to Homework No. 3

SECTION 2.4

Problem 10:
Let $f(t,y) = (t^2+y^2)^{3/2}$. Clearly both f and $\partial f/\partial y$ are continuos on all of the ty plane so that the hypotheses of Theorem 2.4.2 are satisfied everywhere. Note that this does **not** mean that we have existence globally. This subtlety in the Theorem is important.

Problem 12:
Let $f(t,y) = \cot(t)\frac{y}{1+y}$. Since $\cot(t)$ is singular at the values $t = n\pi$ for $n = 0, \pm 1, \pm 2...$, and $\frac{y}{1+y}$ is singular at $y = -1$, then since these are the only points where the hypotheses of Theorem 2.4.2 fail, any connected open set U about (t_0, y_0) whithin the lines $y = -1$ and $t = n\pi$ is valid.

Problem 22:
(a) Let $y_1(t) = 1 - t$. Then $y' = -1$ and

$$\begin{aligned}\frac{-t+(t^2+4y_1)^{1/2}}{2} &= \frac{-t+(t^2+4(1-t))^{1/2}}{2}\\ &= \frac{-t+((t-2)^2)^{1/2}}{2} = -1 \qquad \text{t} \geq 2\end{aligned}$$

so the ODE is satisfied. And since $y_1(2) = -1$ the initial condition is also satisfied. Now let $y_2(t) = -t^2/4$. Then $y' = -t/2$ and

$$\frac{-t+(t^2+4y_2)^{1/2}}{2} = \frac{-t+(t^2-t^2)^{1/2}}{2} = -\frac{t}{2}.$$

and so the ODE is satisfied for y_2 also, and clearly the initial condition is satisfied too.

(b) The existence of these two solutions does not violated Theorem 2.4.2 because the y-derivative of the function $f(t,y) = \frac{-t+(t^2+4y_1)^{1/2}}{2}$ has a singularity at $t^2 + 4y = 0$ and so plugging in $y_1 = 1 - t$ we see that $t^2 + 4(1-t) = 0$ is satisfied when $t = 2$ - which is our initial time t_0. Thus there is **no** open neighborhood about $t = 2$ for which the Theorem holds true which means it is possible for there to exists more than one solution.

(c) Let $y_*(t) = ct + c^2$. Then $y_*' = c$ and

$$\begin{aligned}\frac{-t+(t^2+4y_*)^{1/2}}{2} &= \frac{-t+(t^2+4ct+4c^2)^{1/2}}{2}\\ &= \frac{-t+((t+2c)^2)^{1/2}}{2} = c \qquad \text{t} \geq \text{-2c}\end{aligned}$$

so that the ODE is satisfied. As for the initial condition, notice that $y_*(2) = 4c + c^2$ which satisfies the initial condition provided that $c^2 + 4c = -1$ for which the only solution is $c = -1$. Notice that for this choice of c we recover $y_1(t)$.

Problem 23:
(a) This is a consequence of the linearity of the ODE. Let $y = c\phi(t)$. Clearly since c is a constant then by the product rule we have $y'(t) = (c\phi(t))' = c\phi'(t)$ and $2y = 2c\phi(t)$. Therefore,

$$y' - 2y = c(\phi' - 2\phi)$$

which equals zero since $\phi = e^{2t}$ solves the ODE.

(b) This is a consequence of nonlinearity. Let $\phi(t) = 1/t$. Clearly $\phi'(t) = -1/(t^2) = -(\phi(t))^2$ so that ϕ solves $\phi' + \phi^2 = 0$. Now set $y = c\phi(t)$. Clearly $y'(t) == c/(t^2)$ and $y^2 = c^2/t^2$. Therefore

$$y' + y^2 = -\frac{c}{t^2} + c^2 t^2$$

which equals zero provided $c = 0$ or $c = 1$.

Problem 24:
Again, like part (a) of Problem 23, this result is a consequence of the definition of linearity. Set $y(t) = c\phi(t)$. Then

$$y' + p(t)y = (c\phi(t))' + p(t)(c\phi(t)) = c(\phi' + p(t)\phi)$$

which equals zero since by assumption ϕ solves the ODE.

Problem 25:
More consequences (or definitions) of linearity. Let $y = y_1 + y_2$ as in the book. Then

$$\begin{aligned} y' + p(t)y - g(t) &= y_1' + y_2' + p(t)y_1 + p(t)y_2 - g(t) \\ &= (y_1' + p(t)y_1) + (y_2' + p(t)y_2 - g(t)) \\ &= 0 + 0 \quad \text{by assumption.} \end{aligned}$$

SECTION 2.6

Problem 7:
As always we write the equation as $Mdx + Ndy = 0$ with $M = e^x \sin y - 2y \sin x$ and $N = e^x \cos y + 2\cos x$. Then $M_y = e^x \cos y - 2\sin x$ and $N_x = e^x \cos y - 2\sin x$, which imples $M_y = N_x$. Thus by definition the equation is exact. Thus we may write the ODE as

$$\frac{d}{dx}(e^x \sin y + 2y\cos x) = 0$$

One way to satisfy this equation is to have $y = 0$. Another way is to have y satisfy

$$e^x \sin y + 2y \cos x = c.$$

Problem 23:

Let us look back to the PDE for μ given by Eq.(25) in this section. Divideing that PDE by M and recalling that $(N_x - M_y)/M := Q$ where Q is a function of y **only** we have

$$\mu_y - \frac{N}{M}\mu_x = Q(y)\mu.$$

Now, integrating factors need not be unique, and so we simplify matters by seeking an integrating factor μ that has no x dependence. This makes sense since Q has no x dependence. In this case, $\mu_x = 0$ and so we must solve the ODE

$$\frac{d\mu}{dy} = Q(y)\mu.$$

This is a separable equation for which we may write

$$\frac{d\mu}{\mu} = Q(y)dy$$

and thus by integrating both sides we get

$$\ln(\mu) = \int Q(y)dy.$$

Exponentiating both sides gives us the desired result.

Problem 27:
Again we put the given ODE into familiar form with $M = 1$ and $N = x/y - \sin y$. We then have $M_y = 0$ and $N_x = 1/y$ so that

$$\frac{N_x - M_y}{M} = \frac{1}{y} =: Q(y).$$

Thus we use the result in Problem 23 which says

$$\mu(y) = \exp(\int \frac{dy}{y}) = \exp(\ln(y)) = y.$$

Multiplying our ODE by our integrating factor $\mu = y$ yields

$$ydx + (x - y\sin y)dy = 0$$

which we may write as

$$\frac{d}{dx}(xy + y\cos y - \sin y) = 0.$$

Therefore our solutions are

$$xy + y\cos y - \sin y = c.$$