# Fall 2001, AM33 Solution to hw1

1. Find the general solution to the following problems:

   (a) Problem 1, p38

$$y' + 3y = t + e^{-2t}$$

**Solution**

This is a linear equation with constant coefficients and solved by finding an integrating factor $\mu(t)$. $\mu(t)$ has to satisfy:

$$\mu(t)y' + 3\mu(t)3y = (\mu(t)y)' = \mu(t)y' + \mu'(t)y$$

which means:

$$\mu'(t) = 3\mu(t)$$
$$\frac{\mu'(t)}{\mu(t)} = 3$$
$$\ln(|\mu(t)|)' = 3t$$

we get: $\mu(t) = e^{3t}$ Multiplying both sides of the ODE with this integrating factor we get:

$$y'e^{3t} + 3e^{3t}y = te^{3t} + e^{t}$$
$$(ye^{3t})' = te^{3t} + e^{t}$$

We integrate both sides of this equation with respect to $t$ (The integration of the expression $te^{3t}$ requires integration by parts) and get:

$$ye^{3t} = e^{t} + \frac{te^{3t}}{3} - \frac{e^{3t}}{9} + c$$

   (b) Problem 4, p38

$$y' + (1/t)y = 3\cos 2t \qquad t > 0$$

**Solution**

As in the previous case we need a $\mu(t)$ that satisfies: $\mu'(t) = \frac{\mu(t)}{t}$

$$\frac{\mu'(t)}{\mu(t)} = 1/t$$
$$\ln(|\mu(t)|)' = \ln(t)'$$

So we get $\mu(t) = t$. Thus the ODE becomes:

$$y't + y = 3t\cos 2t \qquad t > 0$$

Integrate both sides of this equation:

$$yt = 3/2(t\sin 2t + \frac{1}{2}\cos 2t) + c$$

or

$$y = \frac{3}{2}\sin 2t + \frac{3}{4t}\cos 2t + c/t$$

(c) Problem 8, p38

$$(1+t^2)y' + 4ty = (1+t^2)^{-2}$$

**Solution** We rewrite the ODE as:

$$y' + \frac{4t}{1+t^2}y = (1+t^2)^{-3}$$

and look for an integrating factor. Proceeding as usual one finds the integrating factor $\mu(t) = (1+t^2)^2$ Thus the ODE becomes:

$$(1+t^2)^2y' + 4t(1+t^2)y = (1+t^2)^{-1}$$

$$(1+t^2)y = \arctan(t) + c$$

$$y = \frac{\arctan(t)}{1+t^2} + \frac{c}{1+t^2}$$

2. Find the solution to the following initial value problems:

(a) Problem 13, p38

$$y' - y = 2te^{2t}, \qquad y(0) = 1$$

**Solution**

The integrating factor is $\mu(t) = e^{-t}$. The equation becomes:

$$e^{-t}y' - e^{-t}y = 2te^{2t}$$

$$(e^{-t}y)' = 2te^{t}$$

Integrating both sides ( right side requires integration by parts)

$$e^{-t}y = 2te^{t} - 2e^{t} + c$$

$$y = 2te^{2t} - 2e^{2t} + ce^{t}$$

To determine $c$ we use the initial value $y(0) = 1$:

$$1 = -2 + c \rightarrow c = 3$$

Hence the solution is:

$$y = 2te^{2t} - 2e^{2t} + 3e^{t}$$

(b) Problem 16, page 38

$$y' + (2/t)y = (\cos t)/t^2 \qquad y(\pi) = 0$$

**Solution**

The integrating factor for this one is : $\mu(t) = t^2$ Thus

$$(yt^2)' = \cos t$$

$$yt^2 = \sin t + c$$

$$y = (\sin t)/t^2 + c/t^2$$

To determine $c$ we use the initial value $y(\pi) = 0$

$$0 = c/\pi^2 \rightarrow c = 0$$

$$y = (\sin t)/t^2$$

(c) Problem 20, page 38

$$ty' + (t+1)y = t, \qquad y(\ln 2) = 1$$

**Solution** We rewrite this ODE to leave $y'$ by itself:

$$y' + \frac{t+1}{t}y = 1$$

and the integrating factor for this equation is : $te^t$. Thus:

$$(te^t y)' = te^t$$

$$y = 1 - 1/t + c/(te^t)$$

We use the initial condition to figure out the constant:

$$1 = 1 - 1/\ln 2 + c/(2\ln 2) \rightarrow c = 2$$

3. Solve problems 28, 30 from section 2.1

(a) Problem 28, p39
Find the value of $y_0$ for which the solution of the initial value problem

$$y' - y = 1 + 3sint, \qquad y(0) = y_0$$

remains finite as $t \to \infty$
**Solution:** One first solves the initial value problem. The solution is:

$$y = -1 - \frac{3}{2}(\sin t + \cos t) + (y_0 + \frac{5}{2})e^t$$

The only term that remains unbounded as $t \to \infty$ is $(y_0 + \frac{5}{2})e^t$. So $y_0 = -5/2$ to make this term 0.

(b) Problem 30, p39 Show that if $a$ and $\lambda$ are positive constants, and b is any real number, then every solution of the equation

$$y' + ay = be^{-\lambda t}$$

has the property that $y \to 0$ as $t \to \infty$
**Solution:**
If one proceeds as suggested in the hint given after the question one finds two solutions:

$$y = (b/(a-\lambda)e^{-\lambda t} + ce^{-at}$$

when $a$ and $\lambda$ are different. And

$$y = (bt + c)e^{-\lambda t}$$

when $a = \lambda$. Both of these solutions converge to 0 as $t \to \infty$.

4. Find general solutions for problems 3,6 from section 2.2

   (a) Problem 3, page 45

$$y' + y^2 \sin x = 0$$

   **Solution**

$$y' + y^2 \sin x = 0$$
$$y' = -y^2 \sin x$$

   Assuming $y$ different from 0 we can rewrite this as:

$$\frac{y'}{y^2} = -\sin x$$
$$\frac{1}{y^2}\frac{dy}{dx} = -\sin x$$

   This is a seperable equation and can be rewritten formally:

$$\frac{dy}{y^2} = -\sin x dx$$

   Integrating both sides we get:

$$-\frac{1}{y} = \cos x + c$$

   or

$$y = \frac{-1}{\cos x + c}$$

   Let's see whether for a constant $c$ $y = c$ satisfies this equation. For $y = c$ $y' = 0$ and $y' + y^2 \sin x = 0 \to y = 0$. Hence $y = 0$ is also a solution.

   (b) Problem 6, page 45

$$xy' = (1 - y^2)^{1/2}$$

   **Solution**

$$xy' = (1 - y^2)^{1/2}$$
$$\frac{y'}{(1 - y^2)^{1/2}} = 1/x$$
$$\frac{1}{(1 - y^2)^{1/2}}\frac{dy}{dx} = 1/x$$

   Or formally:

$$\frac{dy}{(1 - y^2)^{1/2}} = dx/x$$

   Integrating both sides of this equation we get:

$$\arcsin(y) = \ln(|x|) + C$$

   or

$$y = sin(\ln(|x|) + C)$$

   As in the previous example if one explicity looks for a constant solution one finds $y = 1$ and $y = -1$ are also solutions.