Fall 2001, AM33 Solution to hw 10

1. Section 6.3, problem 9

$$f(t) = \begin{cases} t-\pi, & \text{if } \pi \le t \le 2\pi \\ 0, & \text{elsewhere} \end{cases}$$

   f is nonzero only between π and 2π, so we only need to integrate in that region:

$$\begin{aligned} \mathcal{L}\{f\} &= \int_{\pi}^{2\pi} (t-\pi)e^{-st}dt \\ &= \int_{\pi}^{2\pi} te^{-st}dt - \pi\int_{\pi}^{2\pi} e^{-st}dt \\ &= \frac{e^{-\pi s}}{s^2} - \frac{e^{-2\pi s}}{s^2}(1+\pi s) \end{aligned}$$

   Where the first integral is evaluated by using integration by parts.

2. Section 6.3, problem 32

$$f(t) = \sin t, \qquad t \in [0, \pi)$$

   and $f(t+\pi) = f(t)$. By problem 28

$$\begin{aligned} \mathcal{L}\{f\} &= \frac{\int_0^{\pi} e^{-st}\sin t dt}{1-e^{-s\pi}} \\ &= \frac{1+e^{-\pi s}}{(1+s^2)(1-e^{-\pi s})} \end{aligned}$$

   where the integral is evaluated by applying integration by parts twice.

3. Section 6.4, problem 9.

   Method 1:

$$y'' + y = g(t), g(t) = \begin{cases} t/2, \text{if } t \in [0,6) \\ 3, t \ge 6 \end{cases} \quad , y(0) = 0, y'(0) = 1$$

   Take the laplace transform of both sides:

$$\begin{aligned} \mathcal{L}\{y''+y\} &= \mathcal{L}\{g\} \\ \mathcal{L}\{y''\} + \mathcal{L}\{y\} &= \int_0^6 t/2e^{-st}dt + \int_6^{\infty} 3e^{-st}dt \\ \mathcal{L}\{y\}s^2 - sy'(0) - y(0) + \mathcal{L}\{y\} &= \frac{1}{2s^2}(1-e^{-6s}) \\ \mathcal{L}\{y\}(s^2+1) &= \frac{1}{2s^2}(1-e^{-6s}) + 1 \end{aligned}$$

   Thus we have:

$$\mathcal{L}\{y\} = \frac{1}{s^2+1} + (1-e^{-6s})\frac{1}{s^2(s^2+1)}$$

$$\mathcal{L}\{y\} = \frac{1}{s^2+1} + \frac{1}{s^2(s^2+1)} - e^{-6s}\frac{1}{2s^2(s^2+1)}$$

The partial fraction expansion of $\frac{1}{s^2(s^2+1)}$ is $\frac{1}{s^2} - \frac{1}{s^2+1}$. Using this , theorem 6.3.1 on page 311 and the laplace transform table we have that:

$$y(t) = \frac{\sin t + t}{2} - \frac{1}{2} u_6(t)(t - 6 - \sin(t-6))$$

Method 2: Let $f(t) = \frac{t}{2}$. Note that

$$g(t) = f(t) - u_6(t)f(t-6).$$

Therefore,

$$\mathcal{L}\{g\} = (1 - e^{-6s})\mathcal{L}\{f\} = \frac{1}{2s^2}(1 - e^{-6s}).$$

The rest is the same.

4. Section 6.4, problem 10 We are given:

$$y'' + y' + 5/4y = g(t), \quad y(0) = 0, \quad y'(0) = 0, \quad g(t) = \begin{cases} \sin t, \text{if } t \in [0, \pi) \\ 0, t \geq \pi \end{cases}$$

We take the laplace transform of both sides:

$$\begin{aligned} s^2\mathcal{L}\{y\} - sy(0) - y'(0) + s\mathcal{L}\{y\} - y(0) + 5/4\mathcal{L}\{y\} &= \int_0^\pi \sin t e^{-st} dt \\ \mathcal{L}\{y\}(s^2 + s + 5/4) &= \frac{e^{-\pi s} + 1}{1 + s^2} \\ \mathcal{L}\{y\} &= \frac{e^{-\pi s} + 1}{(1 + s^2)(s^2 + s + 5/4)} \\ \mathcal{L}\{y\} &= \frac{e^{-\pi s} + 1}{(1 + s^2)((s + 1/2)^2 + 1)} \end{aligned}$$

The partical fraction expansion of $\frac{1}{(1+s^2)((s+1/2)^2+1)}$ is $\frac{4}{17}(\frac{4s+3}{(s+1/2)^2+1} - \frac{4s-1}{s^2+1})$. Using this, theorem 6.3.1, and the laplace transform table one finds:

$$y = h(t) + u_\pi(t)h(t - \pi) \text{ where } h(t) = \frac{4}{17}(-4\cos t + \sin t + 4e^{-t/2}\cos t + e^{-t/2}\sin t)$$

5. Section 6.5, problem 25 We are given:

$$y'' + 2y' + 2y = f(t); \qquad y(0) = y'(0) = 0$$

The homogeneous part $y'' + 2y' + 2y = 0$ is a constant coefficient problem with chaharacteristic polynomial r^2+2r+2 which have roots: $-1-i, -1+i$. Therefore, we have $y_1 = e^{-t}\cos t$, $y_2 = e^{-t}\sin t$ as a fundamental set of solutions, and it is easy to check that the Wronskian

$$W(y_1, y_2)(t) = e^{-2t}.$$

It follows from Section 3.7, Problem 22 (page 184), that the solution to the original IVP is

$$Y(t) = \int_0^t \frac{e^{-s}\cos s \cdot e^{-t}\sin t - e^{-s}\sin s \cdot e^{-t}\cos t}{e^{-2s}} f(s)\, ds = \int_0^t e^{-(t-s)} f(s)\sin(t-s)ds$$

(b) When $f(s) = \delta(s-\pi)$ the integral becomes:

$$y(t) = \int_0^t e^{-(t-s)}\delta(s-\pi)\sin(t-s)ds$$

For $t < \pi$, the integration region does not include the point $t = \pi$. However, the impulse function $\delta(t-\pi)$ put all the mass on point $t = \pi$ and take value 0 on all $t \neq \pi$. Hence the integral is $y(t) = 0$ for all $t < \pi$.

However, for $t \geq \pi$, the integration region includes point $t = \pi$, hence

$$y(t) = \int_0^t e^{-(t-s)}\delta(s-\pi)\sin(t-s)ds = \int_{-\infty}^{\infty} e^{-(t-s)}\delta(s-\pi)\sin(t-s)ds = e^{-(t-\pi)}\sin(t-\pi).$$

It is now easy to see that

$$y(t) = u_\pi(t)e^{-(t-\pi)}\sin(t-\pi).$$

(c) Now we solve the equation with the laplace transform:

$$\begin{aligned}\mathcal{L}\{y'' + 2y' + 2y\} &= \mathcal{L}\{\delta(t-\pi)\}\\ \mathcal{L}\{y\}s^2 + \mathcal{L}\{y\}2s + 2\mathcal{L}\{y\} &= e^{-s\pi}\\ \mathcal{L}\{y\} &= \frac{e^{-s\pi}}{(s+1)^2+1}\end{aligned}$$

By the laplace transform table and the Theorem 6.3.1 on page 311 we conclude:

$$y(t) = u_\pi(t)e^{-(t-\pi)}\sin(t-\pi)$$

6. Section 6.6, Problem 3 We are asked to see that $\sin t * \sin t$ is not positive for every t.

$$\int_0^t \sin(t-s)\sin(s)ds$$

We can use the trigonometric identity used above: $\sin(t-s) = \sin t\cos s - \cos t \sin s$:

$$\begin{aligned}\int_0^t \sin(t-s)\sin(s)ds &= \int_0^t (\sin t\cos s - \cos t\sin s)\sin s ds\\ &= \sin t\int_0^t \cos s\sin s ds - \cos t\int_0^t \sin^2 s ds\end{aligned}$$

For $t = 2\pi$ the first expression is 0. However the second expression is the $-$ integral of a positive function hence it is less than 0.

7. Section 6.6 Problem 16

$$y'' + 4y' + 4y = g(t), y(0) = 2, y'(0) = -3$$

Let $F(s) = \mathcal{L}\{y\}, G(s) = \mathcal{L}\{g\}$:

$$\begin{aligned}
\mathcal{L}\{y'' + 4y' + 4y\} &= G(s) \\
s^2F(s) - sy(0) - y'(0) + 4sF(s) - 4y(0) + 4F(s) &= G(s) \\
F(s)(s^2 + 4s + 4) &= 2s + 5 + G(s) \\
F(s) &= \frac{2s+5}{s^2+4s+4} + \frac{G(s)}{s^2+4s+4} \\
F(s) &= \frac{2(s+2)}{(s+2)^2} + \frac{1}{(s+2)^2} + \frac{G(s)}{(s+2)^2} \\
F(s) &= \frac{2}{s+2} + \frac{1}{(s+2)^2} + \frac{G(s)}{(s+2)^2}
\end{aligned}$$

Using the laplace transform table and Thm 6.6.1 (the laplace transform of the convolution is equal to the multiplication of the laplace transforms) on page 331 we have:

$$y(t) = 2e^{-2t} + te^{-2t} + \int_0^t (t-s)e^{-2(t-s)}g(s)ds$$

8. Section 6.6 Problem 21 Consider the Volterra integral equation:

$$\phi(t) + \int_0^t (t-s)\phi(s)ds = \sin 2t$$

we are given $u''(t) = \phi(t)$. Substitute this in:

$$u''(t) + \int_0^t (t-s)u''(s)ds = \sin 2t$$

However,

$$\int_0^t (t-s)u''(s)ds = \int_0^t (t-s)d\big(u'(s)\big) = (t-s)u'(s)\big|_0^t - \int_0^t (-1)\cdot u'(s)\,ds = -tu'(0) + u(t) - u(0).$$

It follows that

$$u''(t) + u(t) - tu'(0) - u(0) = \sin 2t$$

(b) We are asked the show that the IVP:

$$u''(t) + u(t) = \sin 2t; u(0) = 0, u'(0) = 0$$

is equivalent to the integral equation. This means we have to show that given a solution to the IVP we can find a solution to the integral equation and vice versa. Assume $u(t)$ is a solution to the IVP. Define $\phi(t) = u''(t)$. By the calculation above $\phi(t)$ is a solution to the integral equation if and only if $u''(t) + u(t) - tu'(0) - u(0) = \sin 2t$. Since we have the initial conditions $u'(0) = u(0) = 0$ this reduces to $u''(t) + u(t) = \sin 2t$ which we know is true because $u(t)$ is a solution to the IVP above.

Conversely assume $\phi(t)$ is a solution to the integral equation. Define $u(t)$ such that $u''(t) = \phi(t)$. By the first calculation we know $u(t)$ satisfies:

$$u''(t) + u(t) - tu'(0) - u(0) = \sin 2t$$

We want $u'(0) = u(0) = 0$ which turns this equation to:

$$u''(t) + u(t) = \sin 2t, u(0) = u'(0) = 0.$$

Hence we are done.

Now we solve the integral equation using the laplace transform:

$$\begin{aligned}
\mathcal{L}\{\phi(t) + \int_0^t (t-s)\phi(s)ds\} &= \mathcal{L}\{\sin 2t\} \\
F(s) + \mathcal{L}\{t\}F(s) &= \frac{2}{s^2+4} \\
F(s) + \frac{1}{s^2}F(s) &= \frac{2}{s^2+4} \\
F(s)\frac{s^2+1}{s^2} &= \frac{2}{s^2+4} \\
F(s) &= \frac{2s^2}{(s^2+4)(s^2+1)}
\end{aligned}$$

The partial fraction expansion of the right handside is: $\frac{2}{3}(\frac{4}{s^2+4} - \frac{1}{s^2+1})$ Using the laplace transform table we find that:

$$\phi(t) = \frac{1}{3}(4\sin 2t - 2\sin t)$$

The initial value problem can be solved using variation of parameters or method of undetermined coefficients it involves nothing new.

9. Extra questions: By the properties of the convolution

$$(f*1)(t) = (1*f)(t) = \int_0^t f(s)ds$$

By associativity of the convolution (i.e$(f*g)*h = f*(g*h)$), and the first part we have:

$$\begin{aligned}
(f*1*1*1.....*1) &= ((((...((f*1)*1)*1)*1)*1)*1).......*1) \\
&= (((...(\int_0^t f(t_n)dt_n)*1)*1)*1).........*1) \\
&= ((..(\int_0^t \int_0^{t_{n-1}} f(t_n)dt_n dt_{n-1})*1)*1).....*1) \\
&\quad ..... \\
&= \int_0^t \int_0^{t_1} \int_0^{t_2} ..... \int_0^{t_{n-2}} \int_0^{t_{n-1}} f(t_n)dt_n dt_{n-1} dt_{n-2}.....dt_2 dt_1
\end{aligned}$$

Let $I(t) = \int_0^t \int_0^{t_1} \int_0^{t_2} ..... \int_0^{t_{n-2}} \int_0^{t_{n-1}} f(t_n) dt_n dt_{n-1} dt_{n-2} ..... dt_2 dt_1$
Now take the laplace transform of both sides:

$$\mathcal{L}\{I(t)\} = \mathcal{L}\{(f * 1 * 1 * 1 .... * 1)(t)\}$$

$$\mathcal{L}\{I(t)\} = \mathcal{L}\{f\} \frac{1}{s} \frac{1}{s} .... \frac{1}{s}$$

$$\mathcal{L}\{I(t)\} = \mathcal{L}\{f\} \frac{1}{s^n}$$

$$\mathcal{L}\{I(t)\} = \mathcal{L}\{(f * \frac{t^{n-1}}{(n-1)!})(t)\}$$

which implies

$$I(t) = (f * \frac{t^{n-1}}{(n-1)!})(t) = (\frac{t^{n-1}}{(n-1)!} * f)(t) = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s) ds$$