Chapter 7. Deterministic Control and Viscosity Solutions

The concept of viscosity solution was introduced by M.G. Crandall and P.L. Lions in their celebrated 1983 paper [3]. A nice survey can be found in [2]. Useful textbooks include [1, 5].

Viscosity solutions are generalized solutions to PDEs. They do not require the differentiability as the classical-sense solutions do. The concept is very useful to characterize the value functions from control problems which usually do not meet the differentiability requirement as a classical solution to the associated Hamilton-Jacobi-Bellman equation.

To illustrate the mains, we will consider *continuous time*, *deterministic* control problems. The associated HJB equation is usually a non-linear first-order PDE.

For example, consider a system with dynamics described by a differential equation

$$dX_t = f(X_t, u_t)\, dt, \quad t \geq 0,$$

with initial state $X_0 = x$. The control $u = \{u_t\}$ ("open-loop" controls) is an arbitrary measurable mapping from $\mathbb{R}_+$ into the *feasible control set* U that is a Borel subset in some Euclidean space. The optimization problem is to find a control policy $\{u_t\}$ so as to minimize the discounted cost

$$J(x; \{u_t\}) \doteq \int_0^\infty e^{-\beta t} c(X_t, u_t)\, dt.$$

The value function is denoted by $v(x)$. The discounting factor β is a positive constant.

The HJB equation associated with this control problem is

$$\beta v(x) - \inf_{u \in U} [\nabla v(x) \cdot f(x, u) + c(x, u)] = 0.$$

In general, the value function does not satisfy this equation in the classical sense.

1 Viscosity solution: definition and basic properties

Let $\Omega \subset \mathbb{R}^d$ be an open domain, and consider the PDE

$$F(x, u(x), \nabla u(x)) = 0, \qquad \forall\, x \in \Omega, \tag{1}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a *continuous* mapping.

Definition 1 A continuous function $u : \Omega \to \mathbb{R}$ is said to be a *viscosity subsolution* of PDE (1) if for any continuously differentiable function $\phi : \Omega \to \mathbb{R}$ and any local maximum point $y \in \Omega$ of $u - \phi$, one has

$$F(y, u(y), \nabla\phi(y)) \leq 0.$$

Similarly, if at any local minimum point $z \in \Omega$ of $u - \phi$, one has

$$F(z, u(z), \nabla\phi(z)) \geq 0,$$

then u is said to be a *viscosity supersolution*.

Finally, if u is both a viscosity subsolution and a viscosity supersolution, then u is called a *viscosity solution*. ■

The following lemma shows that the definition of viscosity solution is consistent with that of classical solutions.

Lemma 1 *A classical solution is a viscosity solution, and a continuously differentiable viscosity solution is a classical solution.*

Proof. Suppose u is a classical solution (i.e., continuously differentiable, and satisfy the equation pointwise). Given any test function ϕ, if $y \in \Omega$ is a local maximum (or local minimum), then $\nabla u(y) = \nabla\phi(y)$. That u is a viscosity solution follows trivially.

Suppose now u is a continuously differentiable viscosity solution. Just pick the test function $\phi \doteq u$. ■

Remark **1** Actually one can show that if u is a viscosity solution and is differentiable at $z \in \Omega$, then $F(z, u(z), \nabla u(z)) = 0$.

Remark **2** The local maximum (or minimum) can be replaced by "strict" local maximum (or minimum) since one can replace ϕ by $\phi(x) + \|x - y\|^2$ at the maximum or minimum point y.

Example 1 The fact that u is a viscosity solution to the PDE $F(x, u, \nabla u) = 0$ does not necessarily imply that it is a viscosity solution to the PDE $-F(x, u, \nabla u) = 0$.

To see this, consider the differential equation

$$-|u'(x)| + 1 = 0, \quad \forall\, x \in (-1, 1).$$

It is not difficult to check that $u(x) = |x|$ is a viscosity solution to this equation.

However, $u(x) = |x|$ is not a viscosity solution to the equation

$$|u'(x)| - 1 = 0.$$

Indeed, consider the test function $\phi(x) \doteq -x^2$. Then $u - \phi = |x| + x^2$ has a local minimum at $z = 0$. But

$$|\phi'(0)| - 1 = -1 < 0.$$

Thus u violates the viscosity supersolution property at $x = 0$. ■

1.1 An equivalent definition of viscosity solutions

Another definition of viscosity solutions uses the concepts of sub- and super-differentials.

Definition 2 For any continuous function $u : \Omega \to \mathbb{R}$ and $x \in \Omega$, the sets

$$\nabla^+ u(x) \doteq \left\{ p \in \mathbb{R}^d : \ \limsup_{y \to x} \frac{u(y) - u(x) - p \cdot (y - x)}{\|y - x\|} \leq 0 \right\}$$

and

$$\nabla^- u(x) \doteq \left\{ p \in \mathbb{R}^d : \ \liminf_{y \to x} \frac{u(y) - u(x) - p \cdot (y - x)}{\|y - x\|} \geq 0 \right\}$$

are called the *super-differential* and *sub-differential* of u at point x.

Lemma 2 *A function u is a viscosity subsolution to (1) if and only if*

$$F(x, u(x), p) \leq 0, \quad \forall\ x \in \Omega,\ p \in \nabla^+ u(x),$$

and u is a viscosity supersolution if and only if

$$F(x, u(x), p) \geq 0, \quad \forall\ x \in \Omega,\ p \in \nabla^- u(x).$$

Proof. We should only prove the equivalence for subsolution, and the other equivalence is totally similar and thus omitted.

It suffices to show that, for any $y \in \Omega$, $p \in \nabla^+ u(y)$ if and only if there exists a continuously differentiable $\phi : \Omega \to \mathbb{R}$ such that $\nabla \phi(y) = p$ and $u - \phi$ has a local maximum at y.

The sufficiency is trivial. For the necessity, assume $p \in \nabla^+ u(y)$. It is not difficult to see that, by assumption, there exists a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $h(0) = 0$ and h is non-decreasing, and satisfies

$$h(r) \geq \sup_{0 < \|x-y\| \leq r} \frac{u(x) - u(y) - p \cdot (x - y)}{\|x - y\|}.$$

Define

$$H(s) \doteq \int_0^s h(r)\, dr$$

and

$$\phi(x) \doteq u(y) + p \cdot (x - y) + H(2\|x - y\|).$$

It is not difficult to check that ϕ is a continuously differentiable function with $\nabla \phi(y) = p$, and furthermore,

$$u(x) - \phi(x) \leq h(\|x - y\|)\|x - y\| - H(2\|x - y\|) \leq 0 = u(y) - \phi(y).$$

This completes the proof. ■

1.2 Some basic properties of viscosity solutions

Below is a list of basic properties of viscosity solutions.

Proposition 1 *1. The maximum of two viscosity subsolutions are again a viscosity subsolution. The minimum of two viscosity supersolutions are again a viscosity supersolution.*

2. Suppose u is a viscosity solution to PDE (1) and $\Phi : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function. If $\Phi'(x) > 0$ for all x, then $v = \Phi(u)$ is a viscosity solution to the equation

$$G(x, v(x), \nabla v(x)) = 0,$$

with

$$G(x, r, p) \doteq F(x, \Phi^{-1}(r), (\Phi^{-1})'(r) \cdot p).$$

3. (Stability) *Let $\{u_n\}$ be viscosity solutions to equations*

$$F_n(x, u_n(x), \nabla u_n(x)) = 0, \qquad \forall\, x \in \Omega.$$

Assume that $u_n \to u$ uniformly on compact sets in Ω and $F_n \to F$ uniformly on compact sets in $\Omega \times \mathbb{R} \times \mathbb{R}^d$. Then u is a viscosity solution to the equation

$$F(x, u(x), \nabla u(x)) = 0.$$

Proof. The proof of the first claim is immediate from the definition and thus omitted.

Proof of Claim 2. We will show that v is a viscosity subsolution and omit the supersolution. Let ϕ be a continuously differentiable function on Ω, and such that $y \in \Omega$ is a local maximum of $v - \phi = \Phi(u) - \phi$. It suffices to show that for $(\Phi^{-1})'[v(y)]\nabla\phi(y) \in \nabla^+ u(y)$, thanks to Lemma 2. Or equivalently,

$$\limsup_{x \to y} \frac{u(x) - u(y) - (\Phi^{-1})'[v(y)]\nabla\phi(y) \cdot (x - y)}{\|x - y\|} \leq 0$$

By construction,

$$v(x) - \phi(x) \leq v(y) - \phi(y), \qquad \forall \;\; x \in N_y,$$

where $N_y \subseteq G$ is an open neighborhood of y. The increasing property of Φ^{-1} implies that

$$\begin{aligned} \Phi^{-1}[v(x)] &\geq \Phi^{-1}\left[v(y) + \phi(x) - \phi(y)\right] \\ &= \Phi^{-1}[v(y)] + (\Phi^{-1})'[v(y)] \cdot [\phi(x) - \phi(y)] + o(|\phi(x) - \phi(y)|) \\ &= \Phi^{-1}[v(y)] + (\Phi^{-1})'[v(y)]\nabla\phi(y) \cdot (x - y) + o(\|x - y\|). \end{aligned}$$

But $\Phi^{-1}(v) = u$, and we complete the proof.

Proof of Claim 3. Again we are only going to give a proof for the case that u is a viscosity subsolution. Let ϕ be a continuously differentiable test function, and $y \in \Omega$ is a local maximum for $u - \phi$. Thanks to Remark 2, we should can assume that y is a strict local maximum. It follows that there exists a $\delta > 0$ such that

$$u(x) - \phi(x) < u(y) - \phi(y), \qquad \forall \; 0 < \|x - y\| \leq \delta.$$

Let y_n be the maximizing point for $u_n - \phi$ in the closed ball $0 \leq \|x - y\| \leq \delta$. Or

$$u_n(x) - \phi(x) < u_n(y_n) - \phi(y_n), \qquad \forall \; 0 < \|x - y\| \leq \delta.$$

The uniform convergence of $\{u_n\}$ implies that any convergent sequence of $\{y_n\}$ converges to y, thus $y_n \to y$. It follows that, at least for n large enough, y_n is a local maximum for $u_n - \phi$. Therefore,

$$F_n(y_n, u_n(y_n), \nabla\phi(y_n)) \leq 0.$$

Letting $n \to \infty$, thanks to the uniform convergence of $\{u_n\}$ and $\{F_n\}$, we have

$$F(y, u(y), \nabla\phi(y)) \leq 0.$$

We complete the proof. ∎

2 The control problem and dynamic programming principle

We consider the infinite horizon problem whose dynamics is described by a differential equation

$$dX_t = f(X_t, u_t)\, dt, \quad t \geq 0,$$

with initial state $X_0 = x$. Sometimes to distinguish between initial condition, we denote the state process X^x. The set of *admissible controls* is the collection of measurable mapping $u = \{u_t\}$ from $\mathbb{R}_+$ into a Borel set U. The optimization problem is to minimize the discounted cost with value function

$$v(x) \doteq \inf_{\{u_t\}} J(x; \{u_t\}) \doteq \inf_{\{u_t\}} \int_0^\infty e^{-\beta t} c(X_t, u_t)\, dt.$$

The discounting factor β is a positive constant.

We will assume the following conditions for the convenience of analysis. They are by no means necessary.

Condition **1**
1. *The set of controls U is compact.*
2. *The function $f : \mathbb{R}^d \times U \to \mathbb{R}^d$ is continuous, and there exists a positive constant L such that*
$$\|f(x,u) - f(y,u)\| \leq L\|x - y\|, \quad \forall\ x, y \in \mathbb{R}^d,\ u \in U.$$
3. *The cost function $c : \mathbb{R}^d \times U \to \mathbb{R}$ is continuous, uniformly bounded, and there exists a constant L such that*
$$|c(x,u) - c(y,u)| \leq L\|x - y\|, \quad \forall\ x, y \in \mathbb{R}^d,\ u \in U.$$

The HJB equation for this control problem is

$$\beta v(x) - \inf_{u \in U} [\nabla v(x) \cdot f(x,u) + c(x,u)] = 0. \tag{2}$$

We have the following main theorem, which characterize the value function as the unique viscosity solution to the HJB equation in a certain class.

Theorem 1 *The value function v is the unique bounded viscosity solution of the HJB equation (2).*

The proof of this result goes as follows: we first establish the *dynamic programming principle (DPP)*, from which it can be shown that the value function is a viscosity solution to the HJB equation. Then we finish the proof by prove the uniqueness by a comparison result.

2.1 The regularity and DPP

We start by giving the regularity of the value function.

Lemma 3 *Assume Condition 1 holds. Then the value function is bounded uniformly continuous.*

Proof. The boundedness of v is trivial. Fix an arbitrary $\varepsilon > 0$. Consider $x, y \in \mathbb{R}^d$, and a control policy u^ε such that

$$v(y) \geq \int_0^\infty e^{-\beta t} c(X_t^y, u_t^\varepsilon)\, dt - \varepsilon.$$

Therefore,

$$v(x) - v(y) \leq \int_0^\infty e^{-\beta t} \left[c(X_t^x, u_t^\varepsilon) - c(X_t^y, u_t^\varepsilon) \right] dt + \varepsilon$$

However, since c is uniformly bounded, there exists a $T > 0$ (independent of x, y) such that

$$\int_T^\infty e^{-\beta t} \left[c(X_t^x, u_t^\varepsilon) - c(X_t^y, u_t^\varepsilon) \right] dt \leq \varepsilon.$$

It follows that

$$v(x) - v(y) \leq L \int_0^T e^{-\beta t} \| X_t^x - X_t^y \| \, dt + 2\varepsilon.$$

However, a standard argument involving the Gronwall's inequality yields,

$$\| X_t^x - X_t^y \| \leq e^{LT} \| x - y \|.$$

Thus, there exists a δ (independent of x, y) such that $v(x) - v(y) \leq 3\varepsilon$ as long as $\|x - y\| \leq \delta$. This completes the proof. ■

Lemma 4 (Dynamic Programming Principle) *Assume Condition 1 holds. Then for every $x \in \mathbb{R}^d$, and $t \geq 0$,*

$$v(x) = \inf_{\{u_t\}} \left[\int_0^t e^{-\beta s} c(X_s, u_s)\, ds + e^{-\beta t} v(X_t) \right].$$

Here X is the state process corresponds to $\{u_t\}$ starting at $X_0 = x$.

Proof. The proof is simple and thus omitted. ■

2.2 The value function as a viscosity solution

In this section we will show that the value function v is a viscosity solution to the HJB equation from DPP.

Lemma 5 *Assume Condition 1 holds. Then the value function v is a viscosity solution to the HJB equation (2).*

Proof. The proof of v being a viscosity subsolution is easier and is given first.

Consider a continuously differentiable test function ϕ, and let y be a local maximum of $v - \phi$. Or there exists a neighborhood of y, say N_y, such that

$$v(x) - v(y) \le \phi(x) - \phi(y), \qquad \forall\, x \in N_y.$$

Consider a control $u_t \equiv \bar{u}$ for some $\bar{u} \in U$. Then DPP implies that

$$v(y) \le \int_0^t e^{-\beta s} c(X_s^y, \bar{u})\, ds + e^{-\beta t} v(X_t^y).$$

It follows that for t small enough,

$$v(X_t^y)(1 - e^{-\beta t}) - \int_0^t e^{-\beta s} c(X_s^y, \bar{u})\, ds \le \phi(X_t^y) - \phi(y).$$

Dividing both sides by t and letting $t \to 0$, since

$$\lim_{t \to 0} \frac{v(X_t^y)(1 - e^{-\beta t})}{t} = \beta v(y),$$

we arrive at

$$\beta v(y) - c(y, \bar{u}) \le \nabla \phi(y) \cdot f(y, \bar{u}).$$

Since $\bar{u}$ is arbitrary, it follows that

$$\beta v(y) - \inf_{u \in U} [\nabla \phi(y) \cdot f(y, u) + c(y, u)] \le 0,$$

and that v is viscosity subsolution.

In order to prove that v is a viscosity supersolution, consider a continuously differentiable test function ϕ and let z be a local minimum of $v - \phi$. Or, there exists a (bounded) neighborhood of z, say N_z, such that

$$v(x) - v(z) \ge \phi(x) - \phi(z), \qquad \forall\, x \in N_z.$$

Thanks to Lemma 6, there exists a $\bar{t} > 0$ such that $X_t^z \in N_z$ for all control u and $0 \le t \le \bar{t}$. Fix an arbitrary $\varepsilon > 0$. For every $0 \le t \le \bar{t}$, there exists a control $u^t = \{u_s^t\}$ such that

$$v(z) \ge \int_0^t e^{-\beta s} c(X_s^z, u_s^t)\, ds + e^{-\beta t} v(X_t^z) - t\varepsilon.$$

It follows that for every $0 \le t \le \bar{t}$,

$$v(X_t^z)(1 - e^{-\beta t}) + t\varepsilon - \int_0^t e^{-\beta s} c(X_s^z, u_s^t)\, ds \ge \phi(X_t^z) - \phi(z).$$

However, we have (why?)

$$\phi(X_t^z) - \phi(z) = \int_0^t \nabla\phi(X_s^z) \cdot f(X_s^z, u_s^t)\, ds = \int_0^t \nabla\phi(z) \cdot f(z, u_s^t)\, ds + o(t)$$

and

$$\int_0^t e^{-\beta s} c(X_s^z, u_s^t)\, ds = \int_0^t c(z, u_s^t)\, ds + o(t),$$

and

$$v(X_t^z)(1 - e^{-\beta t}) = \beta v(z) + o(t).$$

Therefore,

$$\int_0^t \left[\beta v(z) - \nabla\phi(z) \cdot f(z, u_s^t) - c(z, u_s^t)\right]\, ds \ge -t\varepsilon + o(t).$$

It follows that

$$\beta v(z) - \inf_{u \in U} \left[\nabla\phi(z) \cdot f(z, u) + c(z, u)\right] \ge -\varepsilon.$$

Since ε is arbitrary, we completes the proof. ■

Lemma 6 *Assume Condition 1 holds. For every $x \in \mathbb{R}^d$, and every open neighborhood of x, say N_x, there exists a $\bar{t} > 0$ such that for any control $u = \{u_t\}$, the state dynamics given by*

$$dX_t = f(X_t, u_t)\, dt, \qquad X_0 = x$$

satisfies $X_t \in N_x$ for all $0 \le t \le t_0$.

Proof. Given $x \in \mathbb{R}^d$, let

$$M \doteq \sup_{u \in U} \|f(x,u)\|.$$

Since f is continuous and U is compact, M is finite. Since

$$X_t = x + \int_0^t f(X_s, u_s)\, ds = x + \int_0^t f(x, u_s)\, ds + \int_0^t [f(X_s, u_s) - f(x, u_s)]\, ds,$$

we have

$$\|X_t - x\| \le Mt + L \int_0^t \|X_s - x\|\, ds,$$

thanks to Lipschitz continuity. The Gronwall's inequality implies that

$$\|X_t - x\| \le M e^{Lt} t.$$

We completes the proof. ∎

2.3 Uniqueness: a comparison result

The following lemma gives the comparison for viscosity subsoultions and supersolutions. With this we have complete the proof of the main result Theorem 1.

Lemma 7 *Assume Condition 1 holds. Let u_1 be a viscosity subsolution to the HJB equation (2) and u_2 a viscosity supersolution. If both u_1 and u_2 are bounded, then $u_1 \le u_2$.*

Proof. We will prove by contradiction. Suppose that there exists a positive constant $\delta > 0$ and an $z \in \mathbb{R}^d$ such that $u_1(z) - u_2(z) \ge 2\delta$.

Consider now a function $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined by

$$\Phi(x,y) \doteq u_1(x) - u_2(y) - \frac{\|x-y\|^2}{2\varepsilon} - \alpha \left[(1 + \|x\|^2)^p + (1 + \|y\|^2)^p \right],$$

with ε, α, p being some positive constants. The constant α is fixed to be small enough such that

$$\Phi(z,z) = u_1(z) - u_2(z) - 2\alpha(1 + \|z\|^2)^p \ge \delta.$$

The definition of Φ also implies that

$$\lim_{\|x\|, \|y\| \to \infty} \Phi(x,y) \to -\infty.$$

Therefore, there must exist a pair $(\bar{x}, \bar{y})$ that attain the maximum of Φ. In particular,

$$\Phi(\bar{x}, \bar{y}) = \sup_{x,y} \Phi(x, y) \geq \Phi(z, z) \geq \delta.$$

For simplicity, let

$$H(x, p) = \inf_{u \in U} \left[f(x, u) \cdot p + c(x, u) \right].$$

Then the HJB equation can be rewritten as

$$\beta v(x) - H(x, \nabla v(x)) = 0.$$

Define a test function

$$\phi(x) \doteq u_2(\bar{y}) + \frac{\|x - \bar{y}\|^2}{2\varepsilon} + \alpha \left[(1 + \|x\|^2)^p + (1 + \|\bar{y}\|^2)^p \right].$$

Clearly $u_1 - \phi$ attains its maximum at $\bar{x}$, and thus

$$0 \geq \beta u_1(\bar{x}) - H\left[\bar{x}, \nabla \phi(\bar{x})\right]$$

with

$$\nabla \phi(\bar{x}) = \frac{\bar{x} - \bar{y}}{\varepsilon} + 2\alpha p (1 + \|\bar{x}\|^2)^{p-1} \bar{x}.$$

Similarly,

$$0 \leq \beta u_2(\bar{y}) - H[\bar{y}, \nabla \psi(\bar{y})]$$

with

$$\nabla \psi(\bar{y}) = \frac{\bar{x} - \bar{y}}{\varepsilon} - 2\alpha p (1 + \|\bar{y}\|^2)^{p-1} \bar{y}.$$

However, thanks to the Lipschitz continuity and the compactness of U, with $M \doteq \sup\{\|f(0, u)\| : u \in U\}$, we have for every $u \in U$,

$$\begin{aligned}
&[f(\bar{x}, u) \cdot \nabla \phi(\bar{x}) + c(\bar{x}, u)] - [f(\bar{y}, u) \cdot \nabla \psi(\bar{y}) + c(\bar{y}, u)] \\
\leq\ & \frac{L}{\varepsilon} \|\bar{x} - \bar{y}\|^2 + (M + L\|\bar{x}\|) \cdot 2\alpha p (1 + \|\bar{x}\|^2)^{p-1} \|\bar{x}\| \\
&\qquad + (M + L\|\bar{y}\|) \cdot 2\alpha p (1 + \|\bar{y}\|^2)^{p-1} \|\bar{y}\| + L\|\bar{x} - \bar{y}\| \\
\leq\ & \frac{L}{\varepsilon} \|\bar{x} - \bar{y}\|^2 + L\|\bar{x} - \bar{y}\| + (M + 2L)\alpha p \left[(1 + \|\bar{x}\|^2)^p + (1 + \|\bar{y}\|^2)^p \right].
\end{aligned}$$

Fix an p such that $(M + 2L)p \leq \beta$. It follows that

$$\beta[u_1(\bar{x}) - u_2(\bar{y})] \leq \frac{L}{\varepsilon} \|\bar{x} - \bar{y}\|^2 + L\|\bar{x} - \bar{y}\| + \alpha\beta \left[(1 + \|\bar{x}\|^2)^p + (1 + \|\bar{y}\|^2)^p \right].$$

It follows that

$$\beta\delta \leq \beta\Phi(\bar{x}, \bar{y}) \leq \frac{L}{\varepsilon}\|\bar{x} - \bar{y}\|^2 + L\|\bar{x} - \bar{y}\|.$$

Note here the definitions of α and β are independent of ε (but $\bar{x}, \bar{y}$ do), and the above inequality is true for every positive ε.

Observing that $\Phi(\bar{x}, \bar{x}) + \Phi(\bar{y}, \bar{y}) \leq 2\Phi(\bar{x}, \bar{y})$, we have

$$\frac{\|\bar{x} - \bar{y}\|^2}{\varepsilon} \leq [u_1(\bar{x}) - u_1(\bar{y})] + [u_2(\bar{x}) - u_2(\bar{y})]$$

The boundedness of u_1, u_2 implies that as $\varepsilon \to 0$, $\bar{x} - \bar{y} \to 0$. However, it is not difficult to see that there exists a compact set, say $K \subseteq \mathbb{R}^d$ such that $\bar{x}, \bar{y} \in K$ for ε small enough. The uniform continuity of u_1, u_2 in this compact set further implies that

$$\frac{\|\bar{x} - \bar{y}\|^2}{\varepsilon} \to 0.$$

Under these consideration, we have, letting $\varepsilon \to 0$, $\beta\delta \leq 0$, a contradiction. ■

Remark **3** A slight extension of the definition of viscosity solutions is to allow the subsolution to be upper-semicontinuous and the supersolution lower-semicontinuous (the test function ϕ is always continuously differentiable).

Consider the HJB equation (2) with Condition 1 holding. One can again get a similar comparison result such as Lemma 7, with u_1 only be upper-semicontinuous and u_2 only lower-semicontinuous. The proof does not need much modification: Follow the same steps, and we only need to show that $\|\bar{x} - \bar{y}\| \to 0$ and $\varepsilon^{-1}\|\bar{x} - \bar{y}\|^2 \to 0$. But the boundedness of u_1, u_2 again implies that $\|\bar{x} - \bar{y}\| \to 0$ and there exists a compact set $K \subseteq \mathbb{R}^d$ such that $\bar{x}, \bar{y} \in K$ for ε small enough. For this reason, one can assume $\bar{x} \to x^* \in K$ and $\bar{y} \to x^*$ as $\varepsilon \to 0$. However, thanks to the semicontinuities, we have

$$\limsup_{\varepsilon \to 0} \Phi(\bar{x}, \bar{y}) \leq u_1(x^*) - u_2(x^*) - 2\alpha(1 + \|x^*\|^2)^p.$$

But

$$\liminf_{\varepsilon \to 0} \Phi(\bar{x}, \bar{y}) \geq \liminf_{\varepsilon \to 0} \Phi(x^*, x^*) = u_1(x^*) - u_2(x^*) - 2\alpha(1 + \|x^*\|^2)^p.$$

This easily implies that $\varepsilon^{-1}\|\bar{x} - \bar{y}\|^2 \to 0$. ■

2.4 Approximation of value functions and stability

In this section we consider a numerical algorithm for solving the value function v, and prove its convergence. Let h be the discretization parameter, eventually we will let $h \to 0$. We assume that the discretized dynamics (still denoted by X) given by recursion

$$X_{(n+1)h} = X_{nh} + hf(X_{nh}, u_{nh}), \quad X_0 = x,$$

and cost structure

$$v_h(x) \doteq \inf_{\{u_{nh}\}} h \sum_{j=0}^{\infty} (e^{-\beta h})^j c(X_{jh}, u_{jh}).$$

Proposition 2 *Assume that Condition 1 holds. Then $v_h \to v$ uniformly on compact sets as $h \to 0$.*

Proof. Clearly $\{v_h\}$ is continuous and uniformly bounded, and it satisfies the DPE

$$v_h(x) = \inf_{u \in U} \left[hc(x,u) + e^{-\beta h} v_h[x + hf(x,u)] \right].$$

Define

$$\underline{v}(x) \doteq \liminf_{(y,h)\to(x,0)} v_h(y), \qquad \bar{v}(x) \doteq \limsup_{(y,h)\to(x,0)} v_h(y).$$

It is not difficult to see that $\underline{v}$ is lower-semicontinuous and $\bar{v}$ is upper-semicontinuous, and satisfies $\underline{v} \le \bar{v}$.

Suppose for a moment that $\underline{v}$ is a viscosity supersolution and $\bar{v}$ is viscosity subsolution to the HJB equation (2). Then we have $\underline{v} \ge \bar{v}$ (Remark 3). It follows that

$$\bar{v} = \underline{v}$$

and is thus a continuous (and bounded) viscosity solution. Thanks to Theorem 1, we have $v = \bar{v} = \underline{v}$. This implies that $v_h \to v$ uniformly on compact sets. Because otherwise, there exists a compact set $K \subseteq \mathbb{R}^d$ and a positive constant $\delta > 0$ and a sequence $\{x_h \in K\}$ such that

$$|v_h(x_h) - v(x_h)| \ge \delta$$

for all h small enough. Assuming $x_h \to x \in K$, we have either $\bar{v}(x) - v(x) \ge \delta$ or $\underline{v}(x) - v(x) \le -\delta$. A contradiction.

It remains to show that $\underline{v}$ is a viscosity supersolution. The argument for $\bar{v}$ being a subsolution is the similar and thus omitted. Let ϕ be a continuously differentiable test function, and $y \in \mathbb{R}^d$ is a local minimum of $\underline{v} - \phi$. Thanks to Remark 2, we can assume that y indeed is a strict local minimum, say in $\bar{N}_y$ (the closure of a bounded open neighborhood N_y). Assume (if necessary, pick a subsequence) that x_h attains the minimum of $v_h - \phi$ in $\bar{N}_y$, and $x_h \to \bar{x} \in \bar{N}_y$. We must have $\bar{x} = y$ since y is a strict local minimum. Indeed, by definition, we have

$$\underline{v}(\bar{x}) - \phi(\bar{x}) \leq \liminf_{h\to 0} [v_h(x_h) - \phi(x_h)] \leq \limsup_{h\to 0} [v_h(x_h) - \phi(x_h)] \leq \underline{v}(y) - \phi(y),$$

whence $\bar{x} = y$, and $\lim v_h(x_h) = \underline{v}(y)$. The DPE for $\{v_h\}$ and the compactness of U imply that, there exists a $u_h \in U$ that attains the infimum such that

$$v_h(x_h) = e^{-\beta h} v_h[x_h + hf(x_h, u_h)] + hc(x_h, u_h),$$

or equivalently

$$e^{-\beta h}\left[v_h(x_h) - v_h[x_h + hf(x_h, u_h)]\right] + (1 - e^{-\beta h}) v_h(x_h) - hc(x_h, u_h) = 0.$$

It follows that for h small enough,

$$e^{-\beta h}\left[\phi(x_h) - \phi[x_h + hf(x_h, u_h)]\right] + (1 - e^{-\beta h}) v_h(x_h) - hc(x_h, u_h) \geq 0.$$

Without loss of generality, assume that $u_h \to u^* \in U$. Dividing h on both sides and let $h \to 0$, it follows that

$$-\nabla\phi(y) \cdot f(y, u^*) + \beta \underline{v}(y) - c(y, u^*) \geq 0.$$

This complete the proof. ■

3 Finite horizon control problem

Consider a finite horizon control problem, with value function

$$v(x,t) \doteq \inf_{\{u_s\}} \left[\int_t^T e^{-\beta(s-t)} c(X_s, u_s)\, ds + e^{-\beta(T-t)} g(X_T) \right]$$

given $X_t = x$, for all $t \in [0, T]$ and $x \in \mathbb{R}^d$. The dynamics are still described by equation

$$dX_t = f(X_t, u_t)\, dt.$$

The associate HJB equation

$$-v_t + \beta v - \min_{u\in U} \left[\nabla_x v(x,t) \cdot f(x,u) + c(x,u)\right] = 0 \tag{3}$$

with terminal condition

$$v(x, T) = g(x), \qquad \forall\, x \in \mathbb{R}^d. \tag{4}$$

3.1 Definition of viscosity solution

Consider a PDE of type

$$-u_t(x,t) + H(x, u(x,t), \nabla_x u(x,t)) = 0,$$

with terminal condition $u(x,T) = g(x)$.

Definition 3 A continuous function $u : [0,T] \times \mathbb{R}^d$ is said to be the a *viscosity subsolution* if $u(x,T) \leq g(x)$ and that for every continuously differentiable function $\phi : \mathbb{R}^d \times [0,T) \to \mathbb{R}$, and $(x,t) \in \mathbb{R}^d \times [0,T) \to \mathbb{R}$ a local maximum of $u - \phi$, we have

$$-\phi_t(x,t) + H(x, u(x,t), \nabla_x \phi(x,t)) \leq 0.$$

The definition of *viscosity supersolution* is to replace all the "$\leq$" by "$\geq$" and the local maximum by the local minimum.

3.2 The characterization of the value function

We impose the following condition on the terminal payoff $g : \mathbb{R}^d \to \mathbb{R}$.

Condition **2** *The terminal payoff g is bounded and uniformly continuous.*

The following theorem is the main result.

Theorem 2 *Assume $\beta \geq 0$, and assume Condition 1 and Condition 2 hold. Then the value function v is the unique bounded continuous viscosity solution to the HJB equation (3)-(4).*

The proof again is divided into several steps. First, the regularity of the value function and the DPP. Secondly, from DPP we obtain that the value function is a viscosity solution. And finally, from a comparison result we obtain the uniqueness.

Proof. We first show that v is bounded and continuous on $[0,T] \times \mathbb{R}^d$. The boundedness is trivial. As for the continuity, the proof is similar to that of Lemma 3, but more technical due to the time variable. We leave this as an exercise to interested students.

We now prove the DPP. More precisely, we claim that, for every $0 \leq t \leq \bar{t} \leq T$ and every $x \in \mathbb{R}^d$,

$$v(x,t) = \inf_{\{u_t\}} \left[\int_t^{\bar{t}} e^{-\beta(s-t)} c(X_s, u_s)\, ds + e^{-\beta(\bar{t}-t)} v(X_{\bar{t}}, \bar{t}) \right].$$

The proof again is very simple and thus omitted.

We first prove that the value function v is viscosity subsolution. Clearly $v(T,x) = g(x)$. Consider a test function $\phi : [0,T) \times \mathbb{R}^d \to \mathbb{R}$ and such that $(\bar{x}, \bar{t}) \in \mathbb{R}^d \times [0,T)$ is a local maximum of $v - \phi$. In other words, in a small neighborhood of $(\bar{x}, \bar{t})$, say N,

$$v(x,t) - v(\bar{x},\bar{t}) \leq \phi(x,t) - \phi(\bar{x},\bar{t}), \quad \forall\ (x,t) \in N.$$

Fix an arbitrary $u \in U$, and consider the process

$$dX_s = f(X_s, u)dt, \qquad \bar{t} \leq s \leq T,$$

with initial condition $X_{\bar{t}} = \bar{x}$.

For $t \geq \bar{t}$ and close to $\bar{t}$, we have $(t, X_t) \in N$. Moreover, by the DPP, we have

$$v(\bar{x},\bar{t}) \leq \int_{\bar{t}}^{t} e^{-\beta(t-\bar{t})} c(X_s,u)\, ds + e^{-\beta(t-\bar{t})} v(X_t,t).$$

It follows that

$$\phi(\bar{x},\bar{t}) - \phi(X_t,t) \leq \int_{\bar{t}}^{t} e^{-\beta(t-\bar{t})} c(X_s,u)\, ds + \left(e^{-\beta(t-\bar{t})} - 1\right) v(X_t,t).$$

Dividing both sides by $t - \bar{t}$ and then let $t - \bar{t} \downarrow 0$, we have

$$-\phi_t(\bar{x},\bar{t}) - \nabla_x \phi(\bar{x},\bar{t}) \cdot f(\bar{x},u) \leq c(\bar{x},u) - \beta v(\bar{x},\bar{t}).$$

Since u is arbitrary, we have

$$-\phi_t(\bar{x},\bar{t}) + \beta v(\bar{x},\bar{t}) - \inf_{u \in U} \left[\nabla_x \phi(\bar{x},\bar{t}) \cdot f(\bar{x},u) + c(\bar{x},u)\right] \leq 0.$$

As for the viscosity supersolution, again consider the test function ϕ and

$$v(x,t) - v(\bar{x},\bar{t}) \geq \phi(x,t) - \phi(\bar{x},\bar{t}), \quad \forall\ (x,t) \in N.$$

For every $\varepsilon > 0$ and every $t \geq \bar{t}$, there exists a control process $\{u_s\}$ such that

$$v(\bar{x},\bar{t}) \geq \int_{\bar{t}}^{t} e^{-\beta(t-\bar{t})} c(X_s,u_s)\, ds + e^{-\beta(t-\bar{t})} v(X_t,t) - (t-\bar{t})\varepsilon,$$

where $X_{\bar{t}} = \bar{x}$ and

$$dX_s = f(X_s,u_s)\, ds, \qquad \forall\ \bar{t} \leq s \leq T.$$

It follows that for t close to $\bar{t}$,

$$\phi(\bar{x},\bar{t})-\phi(X_t,t)\geq\int_{\bar{t}}^{t}e^{-\beta(t-\bar{t})}c(X_s,u_s)\,ds+\left(e^{-\beta(t-\bar{t})}-1\right)v(X_t,t)-(t-\bar{t})\varepsilon.$$

But

$$\int_{\bar{t}}^{t}e^{-\beta(t-\bar{t})}c(X_s,u_s)\,ds=\int_{\bar{t}}^{t}c(\bar{x},u_s)\,ds+o(t-\bar{t}),$$

and

$$\left(e^{-\beta(t-\bar{t})}-1\right)v(X_t,t)=\beta(t-\bar{t})v(\bar{x},\bar{t})+o(t-\bar{t})$$

and

$$\begin{aligned}\phi(\bar{x},\bar{t})-\phi(X_t,t) &= -\int_{\bar{t}}^{t}\left[\phi_t(X_s,s)+\nabla_x\phi(X_s,s)\cdot f(X_s,u_s)\right]\,ds\\ &= -\int_{\bar{t}}^{t}\left[\phi_t(\bar{x},\bar{t})+\nabla_x\phi(\bar{x},\bar{t})\cdot f(\bar{x},u_s)\right]\,ds+o(t-\bar{t}).\end{aligned}$$

It follows readily that (dividing both sides by $t-\bar{t}$, letting $t\downarrow\bar{t}$ and then $\varepsilon\to 0$), just like Lemma 5,

$$-\phi_t(\bar{x},\bar{t})+\beta v(\bar{x},\bar{t})-\inf_{u\in U}\left[\nabla_x\phi(\bar{x},\bar{t})\cdot f(\bar{x},u)+c(\bar{x},u)\right]\geq 0.$$

It remains to show the uniqueness. Again, we establish a comparison principle; i.e., $u_1\leq u_2$ if u_1 is a bounded viscosity subsolution and u_2 a bounded viscosity supersolution. Consider the function

$$\begin{aligned}\Phi(x,y;t,s) &\doteq u_1(x,t)-u_2(y,s)-\frac{\|x-y\|^2+|t-s|^2}{2\varepsilon}\\ &\quad -\alpha\left[(1+\|x\|^2)^p+(1+\|y\|^2)^p\right]-\eta(T-t+T-s).\end{aligned}$$

Assume that $u_1(z,t_0)-u_2(z,t_0)\geq 2\delta>0$ for some $(z,t_0)\in\mathbb{R}^d\times[0,T)$. Then

$$\sup\Phi\geq\Phi(z,z;t_0,t_0)\geq 2\delta-2\alpha(1+\|z\|^2)^p-2\eta(T-t_0),$$

where the supremum of Φ is taken over all $x,y\in\mathbb{R}^d$ and $t,s\in[0,T]$. Fix α and η and p small enough such that

$$2\eta(T-t_0)\leq\delta/2,\quad p\leq\frac{\eta}{(M+2L)(\|u_1\|_\infty+\|u_2\|_\infty)},\quad 2\alpha(1+\|z\|^2)^p\leq\delta/2,$$

then

$$\sup\Phi\geq\delta.$$

Due to the boundedness of u_1, u_2, it follows easily that the supremum of Φ is attained at, say $(\bar{x}, \bar{y}; \bar{t}, \bar{s})$ for some $\bar{x}, \bar{y} \in \mathbb{R}^d$ and $\bar{t}, \bar{s} \in [0, T]$.

We first want to show that $\bar{t} \neq T$ and $\bar{s} \neq T$ for ε small enough. It is clear that there exists a compact $K \subset \mathbb{R}^d$ such that $\bar{x}, \bar{y} \in K$ for all positive ε. ¿From inequality $\Phi(\bar{x}, \bar{x}; \bar{t}, \bar{t}) + \Phi(\bar{y}, \bar{y}; \bar{s}, \bar{s}) \leq 2\Phi(\bar{x}, \bar{y}; \bar{t}, \bar{s})$ we have

$$\frac{\|\bar{x} - \bar{y}\|^2 + |\bar{t} - \bar{s}|^2}{\varepsilon} \leq [u_1(\bar{x}, \bar{t}) - u_1(\bar{y}, \bar{s})] + [u_2(\bar{x}, \bar{t}) - u_2(\bar{y}, \bar{s})].$$

Therefore, as $\varepsilon \to 0$, we have

$$\|\bar{x} - \bar{y}\| \to 0, \quad |\bar{t} - \bar{s}| \to 0,$$

and (from uniform continuity on $K \times [0, T]$),

$$\frac{\|\bar{x} - \bar{y}\|^2 + |\bar{t} - \bar{s}|^2}{\varepsilon} \to 0.$$

Assume now that $\bar{t} = T$ for a sequence of $\varepsilon \to 0$, then

$$\delta \leq \sup \Phi = \Phi(\bar{x}, \bar{y}; T, \bar{s}) \leq u_1(\bar{x}, T) - u_2(\bar{y}, \bar{s}) \leq u_2(\bar{x}, T) - u_2(\bar{y}, \bar{s}).$$

But the right-hand-side goes to zero along this sequence of ε, a contradiction. Thus $\bar{t} \neq T$. The proof of $\bar{s} \neq T$ is similar.

Now proceed as before, define the test function

$$\begin{aligned} \phi(x, t) \quad &\doteq \quad u_2(\bar{y}, \bar{s}) + \frac{\|x - \bar{y}\|^2 + |t - \bar{s}|^2}{2\varepsilon} + \alpha \left[(1 + \|x\|^2)^p + (1 + \|\bar{y}\|^2)^p \right] \\ &\quad + \eta(T - t + T - \bar{s}). \end{aligned}$$

The function $u_1 - \phi$ attains maximum at $(\bar{x}, \bar{t})$ with $\bar{t} \in [0, T)$. Thus by definition,

$$-\frac{\bar{t} - \bar{s}}{\varepsilon} + \eta + \beta u_1(\bar{x}, \bar{t}) - H\left[\bar{x}, \frac{\bar{x} - \bar{y}}{\varepsilon} + 2\alpha p(1 + \|\bar{x}\|^2)^{p-1}\bar{x}\right] \leq 0.$$

Here

$$H(x, p) \doteq \inf_{u \in U} [p \cdot f(x, u) + c(x, u)]$$

Similarly, we have

$$-\frac{\bar{t} - \bar{s}}{\varepsilon} - \eta + \beta u_2(\bar{y}, \bar{s}) - H\left[\bar{y}, \frac{\bar{x} - \bar{y}}{\varepsilon} - 2\alpha p(1 + \|\bar{y}\|^2)^{p-1}\bar{y}\right] \geq 0.$$

The same argument as in the infinite horizon case leads to

$$\begin{aligned} & 2\eta + \beta[u_1(\bar{x},\bar{t}) - u_2(\bar{y},\bar{s})] \\ \leq \; & \frac{L}{\varepsilon}\|\bar{x}-\bar{y}\|^2 + L\|\bar{x}-\bar{y}\| + (M+2L)\alpha p\left[(1+\|\bar{x}\|^2)^p + (1+\|\bar{y}\|^2)^p\right]. \end{aligned}$$

However, by construction,

$$(M+2L)\alpha p\left[(1+\|\bar{x}\|^2)^p + (1+\|\bar{y}\|^2)^p\right] \leq (M+2L)p(\|u_1\|_\infty + \|u_2\|_\infty) \leq \eta.$$

It follows that

$$2\eta \leq 2\eta + \beta\Phi(\bar{x},\bar{y};\bar{t},\bar{s}) \leq \frac{L}{\varepsilon}\|\bar{x}-\bar{y}\|^2 + L\|\bar{x}-\bar{y}\| + \eta.$$

Letting $\varepsilon \to 0$, the right-hand-side goes to η, a contradiction. ■

3.3 The Hopf-Lax formula

We consider a concrete PDE of type

$$-v_t + H(\nabla_x v) = 0, \quad \forall\, x \in R^d, \quad t \in [0,T)$$

with terminal condition

$$v(x,T) = g(x), \quad \forall\, x \in \mathbb{R}^d.$$

We assume the following condition.

Condition **3** *1. Function $g : \mathbb{R}^d \to \mathbb{R}$ is bounded Lipschitz continuous.*

2. Function H is convex and Lipschitz continuous.

Proposition 3 *Under Condition 3, the unique bounded continuous viscosity solution is given by*

$$v(x,t) = \min_{y\in\mathbb{R}^d}\left[(T-t)L\left(\frac{x-y}{T-t}\right) + g(y)\right], \quad \forall\, x \in \mathbb{R}^d,\ t \in [0,T),$$

where L is the Legendre transform of H, that is

$$L(y) \doteq \sup_{x\in\mathbb{R}^d}\left[x\cdot y - H(x)\right].$$

Proof. For simplicity, let C be the Lipschitz constant for g and H.

It is not difficult to see that v is uniformly bounded. The continuity of v on $\mathbb{R}^d \times [0, T)$ is not so difficult to prove once we observe that L is infinity outside the compact set $\{y : \|y\| \leq C\}$. Indeed, suppose $\|y\| > C$, then for every $k > 0$,

$$L(y) \geq k\|y\|^2 - H(ky) \geq k\|y\|^2 - kC\|y\| - H(0),$$

which implies that, as $k \to \infty$, $L(y) = \infty$.

It remains to show that as $x_n \to x$ and $t_n \to T$, $v(x_n, t_n) \to g(x)$. One direction $\limsup_n v(x_n, t_n) \leq g(x)$ is trivial. For the other direction, it suffices to observe that the minimum is obtained at a point y_n such that $y_n \to y$, and that L is bounded from below.

The proof of the uniqueness is essentially the same as the proof of Theorem 2. We only need to observe that

$$\begin{aligned} H\left[\frac{\bar{x}-\bar{y}}{\varepsilon} + 2\alpha p(1+\|\bar{x}\|^2)^{p-1}\bar{x}\right] \quad &- \quad H\left[\frac{\bar{x}-\bar{y}}{\varepsilon} - 2\alpha p(1+\|\bar{y}\|^2)^{p-1}\bar{y}\right] \\ &\leq \quad C\alpha p\left[(1+\|\bar{x}\|^2)^p + (1+\|\bar{y}\|^2)^p\right], \end{aligned}$$

and we just pick p such that

$$p \leq \frac{\eta}{C(\|u_1\|_\infty + \|u_2\|_\infty)}.$$

We show now that v is a viscosity subsolution. This is achieved by observing an implicit "DPP" connected with definition v. More precisely, we have

$$u(\bar{x}, \bar{t}) = \min_{x \in \mathbb{R}^d}\left[(t - \bar{t})L\left(\frac{\bar{x} - x}{t - \bar{t}}\right) + u(x, t)\right] \tag{5}$$

for every $\bar{t} < t \leq T$ and every $\bar{x} \in \mathbb{R}^d$. The proof of this claim is just a simple convexity argument and thus omitted.

The proof that v is a viscosity solution is quite similar to that of Theorem 2, and is left to the students as an exercise. A useful fact is that

$$H(x) = \sup_{y \in \mathbb{R}^d}\left[x \cdot y - L(y)\right],$$

that is, H is the Legendre transform of L. ■

4 Differential Games

In this section we discuss two-person zero-sum differential games in continuous time on infinite horizon. We will mainly follow the definition of Elliott and Kalton [4]. This definition, involving the so called *strategies*, allows one to use the idea of Dynamic Programming for analysis. It is worth pointing out that in all the definitions of games, a very important aspect is to specify the *information structure* for either of the players.

To be more specific, the state dynamics is determined by the ordinary differential equation

$$dX_t = f(X_t; a_t, b_t)\, dt, \qquad t \geq 0$$

with initial condition $X_0 = x \in \mathbb{R}^d$. Player A picks control $\{a_t\}$ taking values in U and Player B picks $\{b_t\}$ taking values in B, where A and B are some Borel sets in some Euclidean spaces. More precisely, define the sets of "open-loop controls"

$$\begin{aligned} \mathcal{A} &\doteq \{a = (a_t) : \ a \text{ is a measurable function from } [0, \infty) \text{ to } A\} \\ \mathcal{B} &\doteq \{b = (b_t) : \ b \text{ is a measurable function from } [0, \infty) \text{ to } B\}\,. \end{aligned}$$

The associated cost on infinite horizon is

$$J(x; \{a_t\}, \{b_t\}) \doteq \int_0^\infty e^{-\lambda t} c(X_t; a_t, b_t)\, dt$$

where (X_t) is the state process corresponding to initial condition x and control $\{a_t\}$ and $\{b_t\}$. Player A is tries to minimize this cost and is called the *minimizing player* and Player B tries to maximize this cost and is called the *maximizing player*.

Remark **4** A *static* game is where the Player A chooses control from $\mathcal{A}$ and Player B chooses control from set $\mathcal{B}$. The value of the *lower game* is defined as

$$\underline{v}_s \doteq \sup_{b \in \mathcal{B}} \inf_{a \in \mathcal{A}} J(x; \{a_t\}, \{b_t\}),$$

and the value of the *upper game* is defined as

$$\bar{v}_s \doteq \inf_{a \in \mathcal{A}} \sup_{b \in \mathcal{B}} J(x; \{a_t\}, \{b_t\}).$$

By definition, we have $\underline{v}_s \leq \bar{v}_s$. We say the game has a value if $\underline{v}_s = \bar{v}_s$. However, the method of Dynamic Programming is not appropriate for the analysis of such a game. ∎

Definition 4 A *strategy* for Player A is a mapping $\alpha : \mathcal{B} \to \mathcal{A}$ such that, for any $t > 0$, and $b, \bar{b} \in \mathcal{B}$ with $b_s = \bar{b}_s$ for all $0 \le s \le t$, one has $\alpha[b]_s = \alpha[\bar{b}]_s$ for all $0 \le s \le t$. A strategy β for Player B is similarly defined.

Definition 5 Let the collection of strategies for Player A be denoted by Θ and those for Player B by Δ. The *lower value* of the game is defined as

$$\underline{v}(x) \doteq \inf_{\alpha \in \Theta} \sup_{b \in \mathcal{B}} J(x; \alpha[b], b),$$

and the *upper value* of the game as

$$\bar{v}(x) \doteq \sup_{\beta \in \Delta} \inf_{a \in \mathcal{A}} J(x; a, \beta[a]).$$

When $\underline{v}(x) = \bar{v}(x)$ we say the game has a value.

Unlike the static game, it is *not* clear *a priori* that the lower value is dominated by the upper value. It will be proved by a rather indirect approach based on the comparison principle of the viscosity solution.

Example 2 This example shows that the lower value does not always equal the upper value. Consider $f(x; a, b) \doteq (a - b)^2$, and $A = B = [-1, 1]$, with cost (independent of controls)

$$c(x) \doteq \text{sign}(x) \cdot (1 - \exp\{-|x|\}).$$

This running cost is strictly increasing, and the state process $\{X_t\}$ is non-decreasing thanks to the non-negativity of f.

For the lower game, the optimal strategy for Player A is obviously given by $\alpha^*[b](t) \equiv b(t)$, and the lower value of the game is then

$$\underline{v}(x) = c(x).$$

Assume $x \ge 0$. Then $\underline{v}(x) = 1 - \exp\{-x\}$.

As for the upper game, the optimal strategy for Player B is to maximize $f(x; a, b)$, which leads to

$$\beta^*[a](t) = \begin{cases} 1 & ; \quad \text{if } \ a(t) \le 0 \\ -1 & ; \quad \text{if } \ a(t) > 0. \end{cases}$$

For this choice of strategy, the best Player A can do is taking $a(t) \equiv 0$. It follows that

$$\bar{v}(x) = \int_0^\infty e^{-\lambda t} c(x + t)\, dt.$$

Assume $x \geq 0$. Then

$$\bar{v}(x) = \int_0^\infty e^{-\lambda t}(1 - \exp\{-x - t\})\, dt = \frac{1}{\lambda} - \frac{\exp\{-x\}}{1+\lambda}.$$

It can be shown that $\bar{v}(x) > \underline{v}(x)$ and $\bar{v}(x) < \underline{v}(x)$ are both possible dependent on the choice of λ and the initial condition x. ■

4.1 The Dynamic Programming Principle (DPP)

We will adopt the following assumption for the general theory.

Condition **4** *1. Both sets A and B are compact.*

2. The function f is continuous, and there exists a constant L such that

$$\|f(x;a,b) - f(y;a,b)\| \leq L\|x - y\|$$

for all $x \in \mathbb{R}^d$ and all a, b.

3. The cost function c is continuous, uniformly bounded, and satisfies

$$|c(x;a,b) - c(y;a,b)| \leq L\|x - y\|$$

for all $x \in \mathbb{R}^d$ and all a, b.

The first result gives the regularity of the value function, whose proof is very much like that of Lemma 3 and thus omitted.

Lemma 8 *Assume Condition 4 holds. Then both the lower value and the upper value are bounded uniform continuous functions.*

The following Dynamic Programming Principle is essential for the subsequent analysis.

Proposition 4 *Assume Condition 4 holds. Then for all $x \in \mathbb{R}^d$ and $t \geq 0$,*

$$\begin{aligned}
\underline{v}(x) &= \inf_{\alpha \in \Theta} \sup_{b \in \mathcal{B}} \left[\int_0^t e^{-\lambda s} c(X_s; \alpha[b]_s, b_s)\, ds + e^{-\lambda t} \underline{v}(X_t) \right] \\
\bar{v}(x) &= \sup_{\beta \in \Delta} \inf_{a \in \mathcal{A}} \left[\int_0^t e^{-\lambda s} c(X_s; a_s, \beta[a]_s)\, ds + e^{-\lambda t} \bar{v}(X_t) \right].
\end{aligned}$$

Here the state process $\{X_t\}$ is accordingly defined.

Proof. We only give a proof for the part corresponding to the lower game. The case for the upper game is similar and thus omitted.

Denote by $u(x)$ the RHS. We first prove $\underline{v}(x) \leq u(x)$. Fix an arbitrary $\varepsilon > 0$, and pick a strategy $\bar{\alpha} \in \Theta$ such that

$$u(x) \geq \sup_{b \in \mathcal{B}} \left[\int_0^t e^{-\lambda s} c(X_s; \bar{\alpha}[b]_s, b_s)\, ds + e^{-\lambda t} \underline{v}(X_t) \right] - \varepsilon.$$

For each $z \in \mathbb{R}^d$, and let $\alpha_z \in \Theta$ be a strategy such that

$$\underline{v}(z) \geq \sup_{b \in \mathcal{B}} J(z; \alpha_z[b], b) - \varepsilon.$$

Define a new strategy $\hat{\alpha} \in \Theta$ by pasting

$$\hat{\alpha}[b](s) \doteq \begin{cases} \bar{\alpha}[b]_s & ; \quad \text{if } \ 0 \leq s \leq t \\ \alpha_{X_t}[b(\cdot + t)]_{s-t} & ; \quad \text{if } \ s > t \end{cases}$$

It follows that,

$$\begin{aligned} \underline{v}(x) &\leq \sup_{b \in \mathcal{B}} J(x; \hat{\alpha}[b], b) \\ &= \sup_{b \in \mathcal{B}} \left[\int_0^t e^{-\lambda s} c(X_s; \bar{\alpha}[b]_s, b_s)\, ds + \int_t^\infty e^{-\lambda s} c(X_s; \hat{\alpha}[b]_s, b_s)\, ds \right] \\ &\leq \sup_{b \in \mathcal{B}} \left[\int_0^t e^{-\lambda s} c(X_s; \bar{\alpha}[b]_s, b_s)\, ds + e^{-\lambda t} \underline{v}(X_t) + \varepsilon \right] \\ &\leq u(x) + 2\varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ we have $\underline{v}(x) \leq u(x)$.

For the reverse inequality, let $\bar{b} \in \mathcal{B}$ be such that

$$u(x) \leq \int_0^t e^{-\lambda s} c(X_s; \alpha_x[\bar{b}]_s, \bar{b}_s)\, ds + e^{-\lambda t} \underline{v}(X_t) + \varepsilon.$$

However, let $z = X_t$, and define a strategy $\tilde{\alpha} \in \Theta$ defined by

$$\tilde{\alpha}[b]_s = \alpha_x[\tilde{b}]_{s+t},$$

with $\tilde{b}_s \doteq \bar{b}_s$ for $0 \leq s \leq t$ and $\tilde{b}_s \doteq b_{s-t}$ for $s \geq t$. Then there exists an open-loop control $\tilde{b}$ such that

$$\underline{v}(X_t) = v(z) \leq J(z; \tilde{\alpha}[\tilde{b}], \tilde{b}) + \varepsilon.$$

It follows that

$$u(x) \leq J(x; \alpha_x[b^*], b^*) + 2\varepsilon$$

with $b^* \in \mathcal{B}$ defined as

$$b^*_s \doteq \begin{cases} \bar{b}_s & ; \quad \text{if } 0 \leq s \leq t \\ \tilde{b}_{s-t} & ; \quad \text{if } s > t. \end{cases}$$

Thus

$$u(x) \leq J(x; \alpha_x[b^*], b^*) + 2\varepsilon \leq \sup_{b \in \mathcal{B}} J(x; \alpha_x[b], b) + 2\varepsilon \leq v(x) + 3\varepsilon.$$

Letting $\varepsilon \to 0$ we complete the proof. ■

4.2 The characterization of value functions

The following theorem is the main result of the section.

Theorem 3 *Assume Condition 4 holds. Then*

1. *The lower value $\underline{v}$ is the unique bounded continuous viscosity solution to the equation*

$$\lambda v(x) - \sup_{b \in B} \inf_{a \in A} [\nabla v(x) \cdot f(x; a, b) + c(x; a, b)] = 0. \tag{6}$$

2. *The upper value $\bar{v}$ is the unique bounded continuous viscosity solution to the equation*

$$\lambda v(x) - \inf_{a \in A} \sup_{b \in B} [\nabla v(x) \cdot f(x; a, b) + c(x; a, b)] = 0. \tag{7}$$

3. *The lower value is dominated by the upper value; i.e., $\underline{v} \leq \bar{v}$.*

Proof. For notational simplicity, we denote by

$$\underline{H}(x, p) \doteq \sup_{b \in B} \inf_{a \in A} [p \cdot f(x; a, b) + c(x; a, b)] \tag{8}$$

and similarly

$$\bar{H}(x, p) \doteq \inf_{a \in A} \sup_{b \in B} [p \cdot f(x; a, b) + c(x; a, b)] \tag{9}$$

Clearly $\bar{H} \geq \underline{H}$.

Assume for now that claims (1)-(2) (except the uniqueness part) hold. It is not difficult to see that $\bar{v}$ is indeed a supersolution to the equation (6). Thus claim (3) and the uniqueness part will follow from Lemma 8 if one can establish a comparison principle result for bounded continuous viscosity sub- and super- solutions. Indeed, the proof is exactly the same as that of Lemma 7 save that the constant M is now defined as

$$M \doteq \sup\{\|f(0;a,b)\| : \; a \in A, \; b \in B\}.$$

It remains to show that $\underline{v}$ is a viscosity solution to equation (6), or

$$\lambda v - \underline{H}(x, \nabla v) = 0.$$

The proof for $\bar{v}$ is similar and thus omitted. For simplicity, we write $H = \underline{H}$.

We first prove that $\underline{v}$ is a supersolution. Let ϕ be a continuously differentiable test function, and x^* is a local minimum of $\underline{v} - \phi$. We wish to show that

$$\lambda \underline{v}(x^*) - H[x^*, \nabla\phi(x^*)] \geq 0.$$

Assume by contradiction that

$$\lambda \underline{v}(x^*) - H[x^*, \nabla\phi(x^*)] = -4\varepsilon < 0.$$

By the definition of H, there exists a $b^* \in B$ such that

$$\lambda \underline{v}(x^*) - \nabla\phi(x^*) \cdot f(x^*; a, b^*) - c(x^*; a, b^*) \leq -2\varepsilon$$

for every $a \in A$. Due the compactness of A, there exists a small $\bar{t} > 0$ such that

$$\underline{v}(X_t) - \phi(X_t) \geq \underline{v}(x^*) - \phi(x^*)$$

and such that

$$\lambda \underline{v}(X_t) - \nabla\phi(X_t) \cdot f(X_t; \alpha[b^*]_t, b^*) - c(X_t; \alpha[b^*]_t, b^*) \leq -\varepsilon$$

for every $0 \leq t \leq \bar{t}$ and every $\alpha \in \Theta$. Multiplying both sides of the last inequality by $\exp\{-\lambda t\}$ and taking integral, we have

$$\int_0^{\bar{t}} e^{-\lambda t} c(X_t; \alpha[b^*]_t, b^*)\, dt + e^{-\lambda t} v(X_t) \geq \varepsilon(1 - \exp\{-\lambda \bar{t}\})/\lambda + v(x^*).$$

This is true for every $\alpha \in \Theta$, and whence a contradiction to the DPP.

It remains to show that $\underline{v}$ is a subsolution. Let ϕ be a continuously differentiable test function and x^* attains the local maximum of $\underline{v} - \phi$. Again assume by contradiction that

$$\lambda \underline{v}(x^*) - H[x^*, \nabla\phi(x^*)] = 4\varepsilon > 0.$$

Without loss of generality, we can assume $\underline{v}(x^*) = \phi(x^*)$. We claim that there exists a strategy $\alpha^* \in \Theta$ and a $\bar{t} > 0$ such that

$$\int_0^{\bar{t}} e^{-\lambda t} c(X_t; \alpha^*[b]_t, b_t)\, dt + e^{-\lambda t}\phi(X_t) - \phi(x^*) \le -\varepsilon \bar{t}.$$

for every $b \in \mathcal{B}$ and such that

$$\underline{v}(X_t) - \phi(X_t) \le \underline{v}(x^*) - \phi(x^*) = 0, \quad \forall\, 0 \le t \le \bar{t}.$$

Assume this is true for the moment. It follows that

$$\int_0^{\bar{t}} e^{-\lambda t} c(X_t; \alpha^*[b]_t, b_t)\, dt + e^{-\lambda t}\underline{v}(X_t) \le v(x^*) - \varepsilon \bar{t}.$$

This again contradicts the DPP.

We finish the proof by giving the existence of $\alpha^* \in \Theta$. Define

$$\Lambda(x; a, b) \doteq \lambda\phi(x) - \left[\nabla\phi(x) \cdot f(x; a, b) + c(x; a, b)\right].$$

Then by assumption

$$\lambda \underline{v}(x^*) - H[x^*; \nabla\phi(x^*)] = \inf_{b\in B} \sup_{a\in A} \Lambda(x^*; a, b) = 4\varepsilon > 0.$$

Since A, B are both compact and all the relevant functions are continuous, it follows that for every $b \in B$, there exists an $a = a(b) \in A$ such that

$$\Lambda(x^*; a(b), b) \ge 4\varepsilon.$$

Furthermore, there exists an open-ball neighborhood $B(b; r(b))$ with radius $r(b) > 0$ such that

$$\Lambda(x^*; a(b), b') \ge 3\varepsilon, \quad \forall\, b' \in B(b; r(b)).$$

Since B is compact, there exists a finite open cover $\{b_1, b_2, \ldots, b_n\}$ and $\{r_i = r(b_i)\}$ such that

$$B \subseteq \cup_{i=1}^n B(b_i; r_i).$$

Let $a_i = a(b_i)$, then for every $i = 1, \ldots, n$,

$$\Lambda(x^*; a_i, b) \ge 3\varepsilon, \quad \forall\, b \in B(b_i; r_i).$$

Define a measurable mapping $\Phi : B \to A$ as

$$\Phi(b) = a_k, \qquad \text{if } b \in B(b_k; r_k) \setminus \cup_{i=1}^{k-1} B(b_i; r_i),$$

and it follows immediately that

$$\Lambda(x^*; \Phi(b), b) \geq 3\varepsilon, \qquad \forall\ b \in B.$$

Define a strategy (check!)

$$\alpha^*[b]_t \doteq \Phi(b_t).$$

Then

$$\Lambda(x^*; \alpha^*[b]_t, b_t) \geq 3\varepsilon, \qquad \forall\ b \in \mathcal{B}.$$

By the Lipschitz continuity of f and the compactness of A, B, there exists an open neighborhood N_{x^*} such that, for every $x \in N_{x^*}$, we have

$$\Lambda(x; \alpha^*[b]_t, b_t) \geq 2\varepsilon, \qquad \forall\ b \in \mathcal{B}.$$

This N_{x^*} can be chosen small enough such that $\underline{v}(x) - \phi(x) \leq \underline{v}(x^*) - \phi(x^*)$ for every $x \in N_{x^*}$. For the same reason, there exists a $\bar{t} > 0$ such that $X_t \in N_{x^*}$ for every $0 \leq t \leq \bar{t}$ and every $b \in \mathcal{B}$, and such that $1 - \exp\{-\lambda \bar{t}\} \geq 1/2$. Multiplying the last inequality by $\exp\{-\lambda t\}$ and taking integral, we have the desired inequality. ■

Corollary 1 *Assume Condition 4 holds. Suppose that both A and B are convex sets and both functions f and c are convex in a and concave in b. Then the game has a value.*

Proof. Thanks to the uniqueness, it suffices to show that $\underline{H}(x,p) = \bar{H}(x,p)$. But this follows from the classical min/max theorem [6]. ■

Example 3 It is not difficult to show that in general (exercise!),

$$\underline{v}_s \leq \underline{v} \leq \bar{v}_s, \qquad \underline{v}_s \leq \bar{v} \leq \bar{v}_s.$$

However, it is possible that the static game does not have a value (i.e., $\underline{v}_s \neq \bar{v}_s$) even when the dynamic game has a value (i.e., $\underline{v} = \bar{v}$). Consider the following example.

Consider a state process in $\mathbb{R}^4$ with dynamics

$$\begin{aligned} dX_1(t) &= -a(t)X_2(t)\,dt \\ dX_2(t) &= a(t)X_1(t)\,dt \\ dX_3(t) &= -b(t)X_4(t)\,dt \\ dX_4(t) &= b(t)X_3(t)\,dt. \end{aligned}$$

The control sets $A = B = \{-1, 1\}$. The cost is (with $\lambda = 1$)

$$J(x; \{a\}, \{b\}) \doteq \int_0^\infty e^{-t} c(X(t))\, dt$$

with

$$c(x) \doteq \arctan\left[(x_1 - x_3)^2 + (x_2 - x_4)^2\right].$$

It is not difficult to verify that Condition 4 holds in this case. The lower value is then the unique viscosity solution to the equation

$$v(x) - \sup_{b \in B} \inf_{a \in A} [\nabla v \cdot f(x; a, b) + c(x)] = 0.$$

However, it is not difficult to verify that $c(x)$ is a bounded classical solution to this equation. Thus $\underline{v}(x) = c(x)$. Similarly $\bar{v}(x) = c(x)$. In other words, the game has a value, and it equals $c(x)$.

Now consider the static game with initial condition $X(0) = x = (x_1, x_2, x_3, x_4)$. We wish to find some initial conditions x such that

$$\underline{v}_s(x) < \underline{v}(x) = \bar{v}(x) < \bar{v}_s(x).$$

Consider the static lower value, i.e.,

$$\underline{v}_s \doteq \sup_{b \in \mathcal{B}} \inf_{a \in \mathcal{A}} J(x; \{a(t)\}, \{b(t)\}).$$

It follows that

$$\underline{v}_s \leq \sup_{b \in \mathcal{B}} J(x; \{-b(t)\}, \{b(t)\}).$$

Now the RHS is the value function of a control problem, and it follows from Theorem 1 that the RHS is thq unique bounded continuous viscosity solution to the HJB equation

$$v(x) - \sup_{b \in B} [\nabla v \cdot f(x; -b, b) + c(x)] = 0,$$

However, if we plug in the function $\underline{v}(x) = \bar{v}(x) = c(x)$, then

$$\text{LHS} = -\frac{4|x_1 x_4 - x_2 x_3|}{1 + [(x_1 - x_3)^2 + (x_2 - x_4)^2]^2} \leq 0$$

with equality if and only if $x_1 x_4 = x_2 x_3$. A verification argument can easily show that

$$\underline{v}_s(x) < c(x)$$

if $x_1 x_4 \neq x_2 x_3$. Similarly, under this condition, $c(x) < \bar{v}_s(x)$ holds. ■

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