Chapter 3. Optimal Stopping of Markov Chains

The optimal stopping theory is well understood in both continuous time and discrete time. The major development was the work of *Snell Envelop*, which deals with the general non-Markovian setting [3, 1]. The applications of optimal stopping usually specialize to the Markovian setting [2].

The chapter considers the optimal stopping of time-homogeneous Markov chains. Assume that, defined on some probability space (Ω, P) is the Markov chain $\{X_0, X_1, \ldots\}$ taking value in some set S (which is called *state space*). The transition probability kernel is denoted by $P(dy|x)$; i.e.,

$$P(X_{n+1} \in A \,|X_n = x, X_{n-1}, \ldots, X_0) = P(A|x),$$

and

$$E\left[f(X_{n+1})\,|X_n = x, X_{n-1}, \ldots, X_0\right] = \int_S f(y) P(dy|x)$$

for any measurable function $f : S \to R$. The main information from these equalities is that the distribution of X_{n+1}, conditional on the history $\{X_n, X_{n-1}, \ldots, X_0\}$, is independent of $\{X_{n-1}, \ldots, X_0\}$.

Let $c : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ be two functions on the state space. The objective is to minimize the total expected cost

$$E\left[g(X_\tau) + \sum_{j=0}^{\tau-1} c(X_j)\right]$$

over a class of admissible stopping times τ. Depending on the class of stopping times, the problem is classified as finite-horizon and infinite-horizon.

We will assume the following condition throughout the chapter, so that the value function cannot take value plus/minus infinity.

Condition **1** *The function c is non-negative, and g is bounded.*

1 Finite-horizon

Suppose N is a fixed non-negative integer (i.e. *horizon*). Assume the class of admissible stopping times is $\{\tau : P(\tau \leq N) = 1\}$. The value function is denoted by

$$v_N(x) \doteq \inf_{\{\tau : P(\tau \leq N) = 1\}} E\left[g(X_\tau) + \sum_{j=0}^{\tau-1} c(X_j) \,\middle|\, X_0 = x\right]. \tag{1}$$

The DPE for this problem is quite simple: Define

$$\begin{aligned} V_N(x) &\doteq g(x) \\ V_n(x) &\doteq \min\left\{ g(x),\ c(x) + \int_S V_{n+1}(y) P(dy|x) \right\}, \end{aligned}$$

for all $n = 0, 1, \ldots, N-1$. Then one would expect that $v_N(x) = V_0(x)$.

Proposition 1 *The value function $v_N(x) = V_0(x)$, and the optimal stopping time is*

$$\tau_N^* \doteq \inf \{ j \geq 0 : V_j(X_j) = g(X_j) \}.$$

Proof. Consider the process

$$Z_n \doteq \sum_{j=0}^{n-1} c(X_j) + V_n(X_n); \qquad n = 0, 1, \ldots, N.$$

Then the process $\{Z_n : n = 0, 1, \ldots, N\}$ is a sub martingale. Indeed,

$$\begin{aligned} E[Z_{n+1} \,|X_n, X_{n-1}, \ldots, X_0] &= \sum_{j=0}^{n} c(X_j) + E[V_{n+1}(X_{n+1}) \,|X_n] \\ &= \sum_{j=0}^{n} c(X_j) + \int_S V_{n+1}(y) P(dy|X_n) \\ &\geq \sum_{j=0}^{n-1} c(X_j) + V_n(X_n) \\ &= Z_n. \end{aligned}$$

In particular, we have

$$E[Z_\tau] \geq E[Z_0] = V_0(x)$$

for any stopping times taking values in $\{0, 1, \ldots, N\}$. But by definition, $V_j \leq g$ for any j, thus

$$V_0(x) \leq E[Z_\tau] \leq E\left[\sum_{j=0}^{\tau-1} c(X_j) + g(X_\tau) \right].$$

But for every $n = 0, 1, \ldots, N-1$, we have

$$E[Z_{\tau_N^* \wedge (n+1)} \,|X_n, X_{n-1}, \ldots, X_0]$$

$$
\begin{aligned}
&= 1_{\{\tau_N^* \le n\}} Z_{\tau_N^*} + 1_{\{\tau_N^* \ge (n+1)\}} E[Z_{n+1} \,| X_n, X_{n-1}, \ldots, X_0] \\
&= 1_{\{\tau_N^* \le n\}} Z_{\tau_N^*} + 1_{\{\tau_N^* \ge (n+1)\}} \left[\sum_{j=0}^{n} c(X_j) + \int_S V_{n+1}(y) P(dy|X_n) \right] \\
&= 1_{\{\tau_N^* \le n\}} Z_{\tau_N^*} + 1_{\{\tau_N^* \ge (n+1)\}} \left[\sum_{j=0}^{n-1} c(X_j) + V_n(X_n) \right] \\
&= 1_{\{\tau_N^* \le n\}} Z_{\tau_N^*} + 1_{\{\tau_N^* \ge (n+1)\}} Z_n \\
&= Z_{\tau_N^* \wedge n}.
\end{aligned}
$$

Therefore,

$$
V_0(x_0) = E[Z_0] = E[Z_{\tau_N^* \wedge 0}] = E[Z_{\tau_N^* \wedge 1}] = \cdots = E[Z_{\tau_N^* \wedge N}] = E[Z_{\tau_N^*}].
$$

But

$$
E[Z_{\tau_N^*}] = E\left[\sum_{j=0}^{\tau_N^*-1} c(X_j) + V_{\tau_N^*}(X_{\tau_N^*}) \right] = E\left[\sum_{j=0}^{\tau_N^*-1} c(X_j) + g(X_{\tau_N^*}) \right].
$$

It follows that

$$
V_0(x) = v_N(x)
$$

and that τ_N^* is optimal for the problem with horizon N. ■

The following result is a rewrite of the Proposition 1, which is indeed the *forward* form of the above DPE.

Proposition 2 *The value function $\{v_n\}$ satisfies $v_{n+1} \le v_n$ for every n, and*

$$
\begin{aligned}
v_0(x) &= g(x) \\
v_{n+1}(x) &= \min\left\{ g(x),\ c(x) + \int_S v_n(y) P(dy|x) \right\},
\end{aligned}
$$

For each $N \ge 0$, the stopping time

$$
\tau_N^* \doteq \inf\{j \ge 0 : v_{N-j}(X_j) = g(X_j)\}
$$

is optimal for the problem with horizon N.

Example 1 Let us consider a search problem with following setup. Suppose there are totally three offers $\{Y_0, Y_1, Y_2\}$. Offers are iid with common Bernoulli distribution

$$
P(Y_0 = a) = P(Y_0 = b) = 1/2
$$

with $0 < a < b$. Each solicitation will cost c. After the first offer comes in, what should the agent do so as to minimize the total cost plus the minimum price attained?

Solution: The objective is to find

$$v_N(x) \doteq E\left[X_\tau + \sum_{j=0}^{\tau-1} c \,\middle|\, X_0 = x \right],$$

where $N = 2$, and $X_n \doteq Y_0 \wedge \ldots \wedge Y_n$.

It is not difficult to check that $\{X_0, X_1, X_2\}$ is a time-homogeneous Markov chain taking values in $\{a, b\}$, with transition probability matrix

$$\begin{aligned}
P(X_{n+1} = a | X_n = a) &= 1, \\
P(X_{n+1} = b | X_n = a) &= 0, \\
P(X_{n+1} = a | X_n = b) &= 1/2, \\
P(X_{n+1} = b | X_n = b) &= 1/2.
\end{aligned}$$

We will write

$$c = \frac{b-a}{2} + \varepsilon.$$

Define $V_2(x) = x$, or

$$V_2(a) = a, \quad V_2(b) = b.$$

and

$$\begin{aligned}
V_1(a) &= \min\{a, c + 1 \cdot V_2(a) + 0 \cdot V_2(b)\} = a \\
V_1(b) &= \min\{b, c + 1/2 \cdot V_2(a) + 1/2 \cdot V_2(b)\} = b \wedge (b + \varepsilon),
\end{aligned}$$

and

$$\begin{aligned}
V_0(a) &= \min\{a, c + 1 \cdot V_1(a) + 0 \cdot V_1(b)\} = a \\
V_0(b) &= \min\{b, c + 1/2 \cdot V_1(a) + 1/2 \cdot V_1(b)\} = b \wedge (b + 3\varepsilon/2).
\end{aligned}$$

We have $v_2(x) = V_0(x)$ and the optimal stopping time is

$$\tau^* = \inf\{n \geq 0; V_n(x) = x\}.$$

More precisely,

1. For $\varepsilon \geq 0$, $v_2(x) \equiv x$, and the optimal stopping time is $\tau^* = 0$, or stop immediately.

2. For $\varepsilon < 0$, then $v_2(a) = a$, and $v_2(b) = b + 3\varepsilon/2$. It is not difficult to see that

$$\tau^* = \inf\{n \geq 0; X_n = a\}.$$

That is, the optimal policy is to search until the lowest price a is solicited. ■

2 Infinite-horizon

The objective is to determine the value function

$$v(x) \doteq \inf_{\{\tau : P(\tau<\infty)=1\}} E\left[g(X_\tau) + \sum_{j=0}^{\tau-1} c(X_j) \middle| X_0 = x \right]. \tag{2}$$

The restriction of $P(\tau < \infty) = 1$ is to avoid the confusion of the definition of X_τ on set $\{\tau = \infty\}$.

The DPE associated with this problem is

$$V(x) = \min\left\{ g(x),\ c(x) + \int_S V(y)P(dy|x) \right\}. \tag{3}$$

The following result claim that the value function satisfies the DPE.

Theorem 1 *The value function satisfies the DPE (3); i.e.,*

$$v(x) = \min\left\{ g(x),\ c(x) + \int_S v(y)P(dy|x) \right\}.$$

An optimal (finite) stopping time exists if and only if the stopping time

$$\tau^* \doteq \inf\{n \geq 0 : v(X_n) = g(X_n)\}$$

is finite; i.e. $P(\tau^* < \infty) = 1$. *Furthermore, if* τ^* *is finite, then* τ^* *is an optimal stopping time.*

Proof. We claim $v(x) = \lim_n \downarrow v_n(x)$ for every $x \in S$. By definition, we have $v \leq v_n \leq v_{n+1}$ for all n, and whence $v \leq \lim_n v_n$. Now for the reverse inequality, fix $X_0 = x \in S$ and an arbitrary constant $\varepsilon > 0$. There exists a finite stopping time τ_ε such that

$$v(x) + \varepsilon \geq E\left[g(X_{\tau_\varepsilon}) + \sum_{j=0}^{\tau_\varepsilon - 1} c(X_j) \right].$$

But thanks to Condition 1, MCT, and DCT, we have

$$\begin{aligned} E\left[g(X_{\tau_\varepsilon}) + \sum_{j=0}^{\tau_\varepsilon - 1} c(X_j)\right] &= \lim_n E\left[g(X_{\tau_\varepsilon \wedge n}) + \sum_{j=0}^{(\tau_\varepsilon \wedge n)-1} c(X_j)\right] \\ &\geq \lim_n v_n(x). \end{aligned}$$

Therefore,

$$v(x) + \varepsilon \geq \lim_n v_n(x).$$

Since ε is arbitrary, we have $v(x) \geq \lim_n v_n(x)$, which in turn implies $v(x) = \lim_n v_n(x)$, for every $x \in S$.

By Proposition 2, we have

$$v_{n+1}(x) = \min\left\{g(x),\ c(x) + \int_S v_n(y) P(dy|x)\right\}$$

Letting $n \to \infty$ on both sides, and noting that $\{v_n\}$ is clearly uniformly bounded, we have

$$v(x) = \min\left\{g(x),\ c(x) + \int_S v(y) P(dy|x)\right\}$$

from DCT.

Fix $X_0 = x$, and assume now there exists an optimal finite stopping time, say σ. Consider the process

$$Z_n \doteq \sum_{j=0}^{n-1} c(X_j) + v(X_n), \qquad n = 0, 1, \ldots.$$

Then $\{Z_n\}$ is a submartingale, similar to the proof of Proposition 1. We have

$$E[Z_{\sigma \wedge n}] \doteq \left[\sum_{j=0}^{\sigma \wedge n - 1} c(X_j) + v(X_{\sigma \wedge n})\right] \geq E[Z_0] = v(x).$$

Letting $n \to \infty$, thanks to the MCT, DCT, that $v \leq g$, and that σ is optimal, we arrive at

$$v(x) \leq E\left[\sum_{j=0}^{\sigma-1} c(X_j) + v(X_\sigma)\right] \leq E\left[\sum_{j=0}^{\sigma-1} c(X_j) + g(X_\sigma)\right] = v(x).$$

In particular, with probability one,

$$v(X_\sigma) = g(X_\sigma).$$

This implies $\sigma \geq \tau^*$. Whence τ^* is finite if there exists an optimal finite stopping time.

Suppose now τ^* is finite. The same proof in Proposition 1 yields that

$$v(x) = E[Z_0] = \cdots = E[Z_{\tau^* \wedge n}] = \left[\sum_{j=0}^{\tau^* \wedge n-1} c(X_j) + v(X_{\tau^* \wedge n}) \right].$$

Letting $n \to \infty$, we complete the proof, thanks to MCT, DCT, and that $v(X_\tau^*) = g(X_\tau^*)$. ■

Remark **1** For obvious reasons, sometimes the set $\{x \in S : v(x) = g(x)\}$ is called the "stopping region", and the set $\{x \in S : v(x) < g(x)\}$ is called the " continuation region".

Remark **2** The forward form of DPE (Proposition 2) also provide a recursive numerical algorithm to compute v in general.

Corollary 1 *The value function v is* largest *solution to the DPE (3) in the sense that: if u is another solution to the DPE, then $v(x) \geq u(x)$ for every $x \in S$.*

Proof. Suppose u is another solution. The proof of theorem 1 implies that $v(x) = \lim_n v_n(x)$. In order to show $u(x) \leq v(x)$, it suffices to show that $u(x) \leq v_n(x)$ for every n.

We will use induction. For $n = 0$, $u(x) \leq v_0(x) \doteq g(x)$ is trivial. Suppose $u(x) \leq v_n(x)$ for some n. We want to show $u(x) \leq v_{n+1}(x)$. But by Proposition 2,

$$\begin{aligned} v_{n+1}(x) &= \min\left\{ g(x),\ c(x) + \int_S v_n(y) P(dy|x) \right\} \\ &\geq \min\left\{ g(x),\ c(x) + \int_S u(y) P(dy|x) \right\} \\ &= u(x). \end{aligned}$$

This completes the proof. ■

Corollary 2 *Suppose $\{X_0, X_1, \ldots\}$ is a time-homogeneous Markov chain with finite state space: i.e., the state space S is a finite. Then τ^* is always finite and optimal.*

Proof. Let $\bar{x} \in S$ such that

$$g(\bar{x}) = \min_{x \in S} g(x).$$

Since S is a finite set, such a $\bar{x}$ always exists. Clearly, $v(\bar{x}) = g(\bar{x})$. Thus

$$\tau^* = \{n \geq 0 : v(X_n) = g(X_n)\} \leq \{n \geq 0 : X_n = \bar{x}\} \doteq \sigma.$$

But since $\{X_n\}$ is an irreducible, finite Markov chain, σ is finite, whence so is τ^*. ■.

The following proposition is useful to verify whether a solution of the DPE equals the value function.

Proposition 3 *Suppose $\bar{v}$ is a bounded solution to the DPE (3), and that the stopping time*

$$\bar{\tau} \doteq \inf \{n \geq 0 : \bar{v}(X_n) = g(X_n)\}$$

is finite; i.e. $P(\bar{\tau} < \infty) = 1$. Then $\bar{v} = v$ and $\bar{\tau}$ is optimal.

Proof. The proof is exactly the same as the last part of the proof of Theorem 1 with v replaced by $\bar{v}$. We omit the details. ■

Remark **3** Suppose $\beta \in (0, 1)$ is the discount factor, and we consider the discounted optimal stopping problem

$$\inf_{\{\tau : P(\tau < \infty) = 1\}} E\left[\beta^\tau g(X_\tau) + \sum_{j=0}^{\tau-1} \beta^j c(X_j)\right].$$

Then the DPE becomes

$$V(x) = \min\left\{g(x),\ c(x) + \beta \int_S V(y) P(dy|x)\right\}.$$

All the results we have obtained hold for the discounted problem, as long as Condition 1 holds.

3 Examples of infinite horizon

Example 2 *(Concave majorant)* Suppose that $\{X_0, X_1, \ldots\}$ is a simple symmetric random walk on integers $S = \{0, 1, \ldots, b\}$ with absorption at two end points $\{0, b\}$. Let $f : S \to \mathbb{R}$ be a non-negative function. Show that the value function

$$v(x) \doteq \sup_{\{\tau : P(\tau<\infty)=1\}} E\left[\, f(X_\tau) \,|\, X_0 = x\right]$$

is the *least concave majorant* of f (i.e., the smallest concave function that dominates f).

Solution: Theorem 1 implies that the value function v satisfies

$$v(x) = \max\left\{ f(x),\ \int_S v(y) P(dy|x) \right\}, \quad \forall\ x \in S.$$

Thus $v \geq f$. Also, for $x \in \{1, 2, \ldots, b-1\}$, the above equation yields

$$v(x) = \max\left\{ f(x),\ \frac{1}{2} v(x+1) + \frac{1}{2} v(x-1) \right\}.$$

In particular, for all such x,

$$v(x) \geq \frac{1}{2} v(x+1) + \frac{1}{2} v(x-1).$$

Thus v is a concave majorant of f.

It remains to show that v is the smallest concave major ant. To this end, let u be another concave major ant; that is, $u \geq f$ and u is concave. Note that the proof of Theorem 1 implies that $v = \lim_n \downarrow v_n$, where $\{v_n\}$ is recursively determined by

$$\begin{aligned} v_0(x) &= f(x) \\ v_{n+1}(x) &= \max\left\{ f(x),\ \int_S v_n(y) P(dy|x) \right\}. \end{aligned}$$

It suffices to show that $u \geq v_n$ for every n. We will use induction.

The claim holds when $n = 0$. Suppose $u \geq v_n$ for some n. Since u is concave, it is not difficult to see that

$$\int_S u(y) P(dy|x) \leq u(x)$$

for every $x \in S$. Then

$$v_{n+1}(x) \le \max\left\{f(x),\ \int_S u(y)P(dy|x)\right\} \le \max\{f(x), u(x)\} = u(x).$$

We complete the proof. ■

Example 3 *(Discrete put option)* Suppose the log-stock price is modeled by the simple symmetric random process $\{X_0, X_1, \ldots\}$ on $\mathbb{Z}$. In other words, the stock price at time n is $\exp(X_n)$. Let $\beta \in (0,1)$ be a discounter factor. Compute the value function

$$v(x) \doteq \sup_{\{\tau: P(\tau<\infty)=1\}} E\left[\beta^\tau (1-\exp(X_\tau))^+ \Big| X_0 = x\right],$$

and the optimal strategy. The value function represents the price of a discrete put option.

Solution: Theorem 1 and Proposition 3 are applicable since Condition 1 is satisfied; see Remark 3. The DPE associated with this optimal stopping problem is

$$\begin{aligned} V(x) &= \max\left\{(1-e^x)^+,\ \beta\int_{\mathbb{Z}} V(y)P(dy|x)\right\} \\ &= \max\left\{(1-e^x)^+,\ \frac{\beta}{2}V(x+1) + \frac{\beta}{2}V(x-1)\right\}. \end{aligned}$$

The idea is to find a bounded solution to the DPE, and then apply Proposition 3.

It is natural to expect that the optimal stopping time will take form

$$\inf\{n \ge 0 : v(X_n) = (1-\exp(X_n))^+\} = \inf\{n \ge 0 : X_n \le b\}$$

for some integer $b \in \mathbb{Z}$ that has to be determined. Of course, since it is never optimal to stop if $X_n \ge 0$, we also conjecture $b < 0$.

The above discussion prompts us to consider a candidate bounded solution $\bar{v}$ such that $\bar{v}(x) = 1 - e^x$ for all $x \le b$, and $\bar{v}(x) = \beta/2\,[\bar{v}(x+1) + \bar{v}(x-1)]$ for $x > b$. The general solution for the latter difference equation is

$$\bar{v}(x) = Ae^{\alpha x} + Be^{-\alpha x}, \qquad \forall\, x \ge b.$$

for some constants A and B, where

$$\alpha \doteq \log\frac{1-\sqrt{1-\beta^2}}{\beta}. \tag{4}$$

Clearly, $\alpha < 0$. But since we assume $\bar{v}$ is bounded, it follows $B = 0$, and thus

$$\bar{v}(x) = Ae^{\alpha x}, \qquad \forall\ x \geq b.$$

But at $x = b$, we must have

$$1 - e^b = Ae^{\alpha b},$$

or

$$A = e^{-\alpha b}\left(1 - e^b\right).$$

In other words, a candidate solution is

$$\bar{v}(x) = \begin{cases} 1 - e^x & ; \quad \text{if } x \leq b \\ e^{\alpha(x-b)}\left(1 - e^b\right) & ; \quad \text{if } x > b \end{cases} \tag{5}$$

But what is the value of the *free boundary* $b \in \mathbb{Z}$? Recall that for $\bar{v}$ to be a solution, we will have

$$1 - e^x \geq \beta/2[\bar{v}(x+1) + \bar{v}(x-1)], \qquad \forall\ x \leq b, \tag{6}$$

and

$$1 - e^x < \beta/2[\bar{v}(x+1) + \bar{v}(x-1)], \qquad \forall\ x > b. \tag{7}$$

These inequalities indeed uniquely determines $b \in \mathbb{Z}$. We have the following Lemma.

Lemma 1 *There exists a unique negative integer $b \in \mathbb{Z}$ such that the above inequalities (6)-(7) hold. Indeed, $b \in \mathbb{Z}$ is the unique integer such that $b \in (B-1, B]$, with*

$$B = \log \frac{1 - e^{\alpha}}{1 - e^{\alpha - 1}} < 0.$$

The function $\bar{v}$ given by (5) is the value function, and the optimal stopping time is

$$\tau^* = \{n \geq 0 : X_n \leq b\}.$$

The proof of this lemma is given in the Appendix. ■

Remark **4** In the above example, what happens if the discount factor $\beta = 1$? It is not difficult to see that the value function is $v(x) \equiv 1$ for every $x \in \mathbb{Z}$. Indeed, it is clear that $v(x) \leq 1$ for all x. But for an arbitrary $j \in \mathbb{N}$, let $\sigma_j \doteq \inf\{n \geq 0 : X_n \leq -j\}$, then σ_j is finite, and we have, for $x \geq -j$,

$$v(x) = E\left[(1 - \exp(X_{\sigma_j}))^+\right] = (1 - \exp(-j)) \to 1$$

as $j \to \infty$. This implies that $v(x) \geq 1$, and whence $v(x) = 1$. The value function clearly satisfies the DPE

$$v(x) = \max\left\{(1 - e^x)^+,\ [v(x+1) + v(x-1)]/2\right\}.$$

But since the stopping time

$$\inf\{n \geq 0 : v(X_n) = (1 - \exp(X_n))^+\} = \infty,$$

there is no optimal stopping time. Note that $\{\sigma_j\}$ is a sequence of stopping times that approach the value function (i.e., $\{\sigma_j\}$ is an *optimizing sequence*).

4 When Condition 1 is violated: The verification argument

The results in the previous sections rely on Condition 1. What if the condition fails to hold? Or more seriously, what if the cost structure is not as specified; e.g. the cost is path-dependent? In this case, one method we can apply the so-called *verification argument*. This approach has been used many times in Chapter 2, and is also implicitly used in the proof of Theorem 1. The next two examples show that this method is quite general.

Example 4 *(Search for maximum)* This is another search model, but with a different cost. Let $\{Y_0, Y_1, \ldots\}$ be a sequence of iid non-negative random variables, representing the offers, with common distribution F and such that $E[Y_0] < \infty$. Let $c > 0$ be a constant, representing the cost for each solicitation. The objective for the agent is to solve the optimization problem

$$\max_\tau E\left[Y_0 \vee Y_1 \vee \ldots \vee Y_\tau - c\tau\right]$$

over all finite stopping times τ.

Solution: Let $X_n \doteq Y_0 \vee Y_1 \vee \ldots \vee X_n$. Then $\{X_n\}$ is a (non-decreasing) Markov chain. We will denote its transition probability matrix by $P(dy|x)$. Moreover, the optimization problem is equivalent to

$$\inf_\tau E\left[-X_\tau + \sum_{j=0}^{\tau-1} c\right].$$

This does not satisfies Condition 1 since $\{X_n\}$ may not be bounded. Denote the value function by

$$v(x) = \sup_{\{\tau : P(\tau<\infty)=1\}} E\left[X_\tau - c\tau \,|\, X_0 = x\right].$$

We will consider two cases separately.

1. If $E[Y_0] \leq c$, then the optimal stopping time is $\tau^* = 0$, and the value function is $v(x) \equiv x$ for all x. Indeed, it is trivial to check that the process $\{X_n - cn\}$ is a supermartingale. Thus, for every finite stopping time τ and $n \in \mathbb{N}$, we have

$$E\left[X_{\tau\wedge n} - c(\tau \wedge n)\right] \leq E[X_0] = x$$

Letting $n \to \infty$, using MCT twice, we have

$$E\left[X_\tau - c\tau\right] \leq x.$$

Thus $v(x) \leq x$. But if we take $\tau^* = 0$, then $E\left[X_{\tau^*} - c\tau^*\right] = x$. Hence $v(x) = x$ and $\tau^* = 0$ is optimal.

2. Suppose now $E[Y_0] > c$. We claim that the value function is

$$v(x) = \begin{cases} x^* & ; \quad \text{if } \ x \leq x^* \\ x & ; \quad \text{if } \ x > x^* \end{cases} \tag{8}$$

where x^* is the solution to the equation

$$c = \int_{x^*}^{\infty} [1 - F(y)]\, dy.$$

We also assert that the optimal stopping time is

$$\tau^* \doteq \inf\{n \geq 0 : X_n \geq x^*\} = \inf\{n \geq 0 : Y_n \geq x^*\}.$$

Before proving the claim, it worth pointing out that x^* always exists, since

$$\int_0^{\infty} [1 - F(y)]\, dy = E[Y_0] > c,$$

and that τ^* is obviously finite.

The idea of the proof is the same as Case 1. Denote by $\bar{v}$ the RHS of equation (8). It is not difficult to check that the

$$\begin{aligned} \bar{v}(x) &= \max\left\{x, \ -c + \int \bar{v}(y) P(dy|x)\right\} \\ &= \max\left\{x, \ -c + \int \bar{v}(x \vee y)\, dF(y)\right\} \end{aligned} \tag{9}$$

Indeed, for $x > x^*$, we have

$$
\begin{aligned}
& -c + \int \bar{v}(x \vee y)\, dF(y) \\
&= -c + \int_0^x x\, dF(y) + \int_x^\infty y\, dF(y) \\
&= -c + E[Y_0] + \int_0^x (x - y)\, dF(y) \\
&= -c + E[Y_0] + \int_0^x F(y)\, dy \\
&= -\int_{x^*}^\infty [1 - F(y)]\, dy + \int_0^\infty [1 - F(y)]\, dy + \int_0^x F(y)\, dy \\
&= \int_0^{x^*} [1 - F(y)]\, dy + \int_0^x F(y)\, dy \\
&= x - \int_{x^*}^x [1 - F(y)]\, dy,
\end{aligned}
$$

and thus RHS $= x =$ LHS. For $x \leq x^*$, we have similarly

$$
-c + \int \bar{v}(x \vee y)\, dF(y) = -c + \int_0^{x^*} x^*\, dF(y) + \int_{x^*}^\infty y\, dF(y) = x^*,
$$

and thus again RHS $= x^* =$ LHS.

It follows equation (9) that the process $\{\bar{v}(X_n) - cn\}$ is a supermartingale. Thus for every finite stopping time τ and $n \in \mathbb{N}$, we have

$$
E\left[\bar{v}(X_{\tau \wedge n}) - c(\tau \wedge n)\right] \leq \bar{v}(x).
$$

Letting $n \to \infty$, using MCT twice, then taking supremum over τ on the LHS, and observing $\bar{v}(x) \geq x$, we arrive at $v(x) \leq \bar{v}(x)$. But for the stopping time τ^*, one can show analogously, for every $n \geq 0$,

$$
\begin{aligned}
E\left[\bar{v}(X_{\tau^* \wedge (n+1)}) - c(\tau^* \wedge (n+1))\right] &= E\left[\bar{v}(X_{\tau^* \wedge n}) - c(\tau^* \wedge n)\right] \\
&= \cdots = \bar{v}(x).
\end{aligned}
$$

Letting $n \to \infty$ and using MCT twice, we have

$$
\bar{v}(x) = E\left[\bar{v}(X_{\tau^*}) - c\tau^*\right] = E\left[X_{\tau^*} - c\tau^*\right].
$$

Thus $\bar{v}(x) = v(x)$ and τ^* is optimal. ■

Example 5 *(Discrete up-and-out put option)* Suppose the log-stock price is modeled by the simple symmetric random process $\{X_0, X_1, \ldots\}$ on $\mathbb{Z}$. Let $\beta \in (0,1)$ be a discounter factor. Compute the value function

$$v(x) \doteq \sup_{\{\tau: P(\tau<\infty)=1\}} E\left[\beta^\tau \left(1-\exp(X_\tau)\right)^+ \cdot 1_{\{\max_{0\le j\le \tau} X_j < H\}} \middle| X_0 = x\right].$$

Here $H \in \mathbb{Z}$ is a positive integer. The value function represents the price of a discrete up-and-out put option with barrier H.

Consider the following *variational inequality*, which is the DPE in detail.

Variational Inequality: Find a bounded function $V : Z \to \mathbb{R}^+$ and an integer (free boundary) $b < 0$, such that

$$\begin{aligned}
V(x) &= 0, & x \ge H \quad (10)\\
V(x) &= \beta\left[V(x+1) + V(x-1)\right]/2; & b < x < H \quad (11)\\
V(x) &= 1 - e^x, & x \le b \quad (12)\\
V(x) &> (1-e^x)^+, & b < x < H \quad (13)\\
V(x) &\ge \beta\left[V(x+1) + V(x-1)\right]/2. & x \le b \quad (14)
\end{aligned}$$

Lemma 2 *Suppose (V, b) is a solution to the variational inequality. Then $v(x) = V(x)$ for all x and the optimal stopping time is*

$$\tau^* \doteq \inf\{n \ge 0 : X_n \le b\}.$$

Proof. Let $\sigma \doteq \inf\{n \ge 0 : X_n \ge H\}$. Then the value function can be rewritten as

$$v(x) = \sup_{\{\tau: P(\tau<\infty)=1\}} E\left[\beta^\tau \left(1-\exp(X_\tau)\right)^+ \cdot 1_{\{\tau<\sigma\}} \middle| X_0 = x\right].$$

Now consider the process $Z_n \doteq \{\beta^{\sigma\wedge n} V(X_{\sigma\wedge n}) : n = 0, 1, \ldots\}$. We claim $\{Z_n\}$ is a supermartingale. Indeed,

$$\begin{aligned}
&E[Z_{n+1} \,|X_n, \ldots, X_0]\\
&\quad= \beta^\sigma V(X_\sigma) 1_{\{\sigma\le n\}} + 1_{\{\sigma\ge n+1\}} \beta^{n+1} E[V(X_{n+1}) \,|X_n, \ldots, X_0]\\
&\quad= \beta^\sigma V(X_\sigma) 1_{\{\sigma\le n\}} + 1_{\{\sigma\ge n+1\}} \beta^{n+1} \left[V(X_n+1) + V(X_n-1)\right]/2\\
&\quad\le \beta^\sigma V(X_\sigma) 1_{\{\sigma\le n\}} + 1_{\{\sigma\ge n+1\}} \beta^{n} V(X_n)\\
&\quad= Z_n.
\end{aligned}$$

It follows from Optional Sampling Theorem that for any stopping time τ and any $n \in N$, we have

$$E[Z_{\tau \wedge n}] \leq E[Z_0] = V(x).$$

Letting $n \to \infty$, the DCT further implies that

$$E[Z_\tau] = \lim_n E[Z_{\tau \wedge n}] \leq V(x).$$

But

$$Z_\tau = \beta^{\sigma \wedge \tau} V(X_{\sigma \wedge \tau}) \geq \beta^\tau \left(1 - \exp(X_\tau)\right)^+ \cdot 1_{\{\tau < \sigma\}}.$$

Whence

$$V(x) \geq E\left[\beta^\tau \left(1 - \exp(X_\tau)\right)^+ \cdot 1_{\{\tau < \sigma\}}\right].$$

Taking supremum over τ on the RHS, we have $V(x) \geq v(x)$.

Now define $Z_n^* \doteq \{\beta^{\sigma \wedge \tau^* \wedge n} V(X_{\sigma \wedge \tau^* \wedge n}) : n = 0, 1, \ldots\}$. Similarly to the above argument, we have

$$\begin{aligned}
&E[Z_{n+1}^* \,|X_n, \ldots, X_0] \\
&= \beta^{\sigma \wedge \tau^*} V(X_{\sigma \wedge \tau^*}) 1_{\{\sigma \wedge \tau^* \leq n\}} + 1_{\{\sigma \wedge \tau^* \geq n+1\}} \beta^{n+1} \left[V(X_n + 1) + V(X_n - 1)\right]/2 \\
&= \beta^{\sigma \wedge \tau^*} V(X_{\sigma \wedge \tau^*}) 1_{\{\sigma \wedge \tau^* \leq n\}} + 1_{\{\sigma \wedge \tau^* \geq n+1\}} \beta^n V(X_n) \\
&= Z_n^*.
\end{aligned}$$

In particular,

$$E\left[\beta^{\sigma \wedge \tau^* \wedge n} V(X_{\sigma \wedge \tau^* \wedge n})\right] = E[Z_n^*] = E[Z_{n-1}^*] = \ldots = E[Z_0] = V(x).$$

Letting $n \to \infty$, we have, thanks to DCT,

$$E\left[\beta^{\sigma \wedge \tau^*} V(X_{\sigma \wedge \tau^*})\right] = V(x).$$

But

$$\beta^{\sigma \wedge \tau^*} V(X_{\sigma \wedge \tau^*}) = \beta^{\tau^*} \left(1 - \exp(X_{\tau^*})\right)^+ \cdot 1_{\{\tau^* < \sigma\}},$$

Thus $V(x) \geq v(x)$, which in turn implies that $V(x) = v(x)$ and that τ^* is optimal. ■

Lemma 3 *The solution to the variational inequality is*

$$V(x) = \begin{cases} 0 & ; \quad \text{if } x \geq H \\ Ae^{\alpha x} + Be^{-\alpha x} & ; \quad \text{if } b < x < H \\ 1 - e^x & ; \quad \text{if } x \leq b \end{cases},$$

with α as defined in (4), and

$$A = \frac{(1-e^b)e^{-\alpha b}}{1-e^{2\alpha(H-b)}}, \quad B = \frac{(1-e^b)e^{\alpha b}}{1-e^{-2\alpha(H-b)}}.$$

The free boundary is the unique integer in the interval $(D-1, D]$, with D being the unique negative solution satisfying the equation

$$(1-e^{D-1})\left[1-e^{2\alpha(H-D)}\right] + (e^D - 1)\left[e^{-\alpha} - e^{\alpha+2\alpha(H-D)}\right] = 0.$$

Proof. The proof of the lemma is just some technical details, and is deferred to the Appendix. ■

A Appendix

Proof of Lemma 1. It suffices to solve for an integer $b < 0$. The rest is implied by Proposition 3. The inequalities (6)-(7) amount to

$$\begin{aligned} 1-e^x &\geq \beta/2\left(1-e^{x+1}+1-e^{x-1}\right), && \forall\, x \leq b-1 \\ 1-e^b &\geq \beta/2\left(e^{\alpha}\left(1-e^b\right)+1-e^{b-1}\right), && x = b \\ (1-e^x)^+ &< e^{\alpha(x-b)}\left(1-e^b\right), && \forall\, x \geq b+1. \end{aligned}$$

The last inequality is equivalent to

$$e^x + e^{\alpha(x-b)}\left(1-e^b\right) > 1, \quad \forall\, x \geq b+1$$

However, since $b < 0$, the LHS is a convex function, and taking value 0 if $x = b$. Thus the above inequality reduces to

$$e^{b+1} + e^{\alpha}(1-e^b) > 1,$$

or

$$b > \log\frac{1-e^{\alpha}}{e-e^{\alpha}} = B - 1.$$

The inequality corresponding to $x = b$ gives

$$e^b\left(1-\beta e^{\alpha}/2-\beta e/2\right) \leq 1-\beta e^{\alpha}/2-\beta/2.$$

But observing $\exp(\alpha)+\exp(-\alpha) = 2/\beta$, we have

$$b \leq \log\frac{e^{-\alpha}-1}{e^{-\alpha}-e^{-1}} = B.$$

These uniquely determine b as the integer in the interval $(B-1, B]$.

It remains to verify the inequality for $x \leq b-1$, or

$$1-\beta \geq e^{x}\left(1-\beta e/2-\beta e^{-1}/2\right), \quad \forall\, x \leq b-1.$$

But this is trivial since $e+e^{-1}>2$ and

$$\text{RHS} \leq e^{b-1}(1-\beta) < 1-\beta.$$

We finish the proof. ■

Proof of Lemma 3. Consider the function

$$f(y) \doteq (1-e^{y-1})\left[1-e^{2\alpha(H-y)}\right] + (e^{y}-1)\left[e^{-\alpha}-e^{\alpha+2\alpha(H-y)}\right]$$

for all $y \leq 0$. It can be shown that f is a strictly increasing function with $f(\infty) < 0 < f(0)$ (the details are left for interested students). Thus there exists a unique solution $D < 0$ such that $f(D) = 0$. The monotonicity of f implies that $f(y) \geq 0$ for $y \geq D$, and $f(y) < 0$ for $y < D$.

Let b be the unique integer such that $b \in (D-1, D]$. We want to show that (V, b) defines a solution to the variational inequality. Clearly V is bounded. It remains to show that inequalities (13)-(14). We first show inequality (13). It is not difficult to see that, for $b < x < H$,

$$V(x) = e^{-\alpha x}\left(Ae^{2\alpha x}+B\right) \geq e^{-\alpha x}\left(Ae^{2\alpha b}+B\right) = e^{-\alpha(x-b)} > 0.$$

Thus, we only need to show $V(x) > 1-e^{x}$ for all $b < x < H$. However,

$$(Ae^{\alpha x}+Be^{-\alpha x}) = \alpha^{2}(Ae^{\alpha x}+Be^{-\alpha x}) > 0$$

on interval (b, H). Therefore, V is convex and so is $V(x)+e^{x}$. Since $V(b)+e^{b}=1$, inequality (13) amounts to $V(b+1) > 1-e^{b+1}$, or equivalently (after some algebra) $f(b+1) > 0$, or $b > D-1$. But by definition, this inequality holds, so does (13). Next we show inequality (14). But for $x < b$, it is equivalent to

$$1-\beta \geq e^{x}(1-\beta e/2-\beta e^{-1}/2).$$

This is trivial for $x < b < 0$. We remains to show, for $x = b$,

$$1-e^{b} \geq \beta/2\left(Ae^{\alpha(b+1)}+Be^{-\alpha(b+1)}+1-e^{b-1}\right).$$

This, after some algebra, is equivalent to $f(b) \leq 0$, or $b \leq D$. We complete the proof. ■

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