Homework 2

1. This exercise gives an example where the optimal policy is necessarily non-stationary. Consider an MCP $\{X_n\}$ with state space $S = \{0, 1\}$. The control set $U(0) = [0, \infty)$ and $U(1) = \{0\}$ (i.e., state "1" is uncontrolled). The dynamics of the system is

$$P(X_{n+1} = 1 \,|\, X_n = 1, u_n = u) = 1, \qquad \forall\, u \in U(1),$$

and for every $u \in U(0)$,

$$P(X_{n+1} = 1 \,|\, X_n = 0, u_n = u) = \frac{1}{2} = P(X_{n+1} = 0 \,|\, X_n = 0, u_n = u).$$

The optimization problem is

$$v(x) \doteq \sup_{\{u_n\}} E_x \sum_{j=0}^{\infty} \beta^j u_j = \sup_{\{u_n\}} J(x; \{u_n\}).$$

Show that $v(1) = 0$ and $v(0) = +\infty$. Also show that for any stationary policy $\phi : x \mapsto U(x)$, the corresponding cost

$$J(0; \phi) = \frac{2}{2 - \beta}\phi(0), \qquad J(1; \phi) = 0.$$

So no stationary policy is optimal. Argue that the non-stationary policy $u_n^* \doteq (2/\beta)^n$ if $x_n = 0$ is an optimal policy.

2. Consider the utility maximization problem with

$$v(x) = \sup_{\{u_n\}} E_x \sum_{j=0}^{\infty} \beta^j F(u_j)$$

where the dynamics are determined by

$$X_{n+1} = R_{n+1}(X_n - u_n), \qquad 0 \le u_n \le X_n.$$

The utility function $F(u) \doteq u^\alpha$ for some constant $\alpha \in (0, 1)$. The random variable $\{R_n\}$ is an iid sequence of non-negative random variables such that $\beta E[R_n^\alpha] < 1$. Solve this problem and find the optimal policy.

3. Consider an MCP $\{X_n\}$ with finite state space and finite control set $U(x)$ for every $x \in S$. The running cost $c(X_n, u_n)$ is assumed to be non-negative. Let V_β be the value function for the infinite horizon problem; i.e.,

$$V_\beta(x) \doteq \inf_{\{u_n\}} E_x \sum_{j=0}^{\infty} \beta^j c(X_j, u_j),$$

and $\Lambda(x)$ the value function for the long-run average cost problem. Show that if there is a state $z \in S$ and a constant L such that

$$|V_\beta(x) - V_\beta(z)| \leq L, \qquad \forall\, x \in S, \beta \in (0,1),$$

then $\Lambda(x) \equiv \lambda$ for some constant λ.

4. Use the result from the previous exercise to solve the following problem. Consider a machine that can be in any one of the states $S = \{1, 2, \ldots, K\}$. The options at the start of each time period is to

   (a) let the machine operate one more period in the state it currently is, with a cost $g(x)$ where x is the current state. At the beginning of next period, the state of machine becomes y with probability P_{xy}.

   (b) repair the machine at a positive constant cost R and bring it back to state 1, and then let the machine operate in state "1" (for this period) with cost $g(1)$. At the beginning of next period, the state of machine becomes y with probability P_{1y}.

   The cost function $g : S \to \mathbb{R}$ is assumed to be non-negative and strictly increasing, and that the probability transition matrix P is monotone, that is, for every non-decreasing function $f : S \to \mathbb{R}$, the function

$$Pf(x) \doteq \sum_{y \in S} P_{xy} f(y)$$

   is also non-decreasing in x.

   Show that the long-run average problem has a value function $\Lambda(x) \equiv \lambda$ for some constant λ. Can you find an optimal policy?

5. Consider an industry where the demand $\{Z_n\}$ follows a time homogeneous Markov chain with transition probability function P. The price is determined by the demand and the supply. More precisely, if q is

the supply and z is the demand then $p = D(q, z)$ is the price. The "consumer's surplus" is defined as

$$U(q, z) \doteq \int_0^q D(\nu, z)\, d\nu, \qquad \forall\, q \geq 0.$$

Let $\{X_n\}$ denote the total industry capital stock at time n. $\{X_n\}$ are assumed to be strictly positive. Assume the size of supply equals the size of capital stock.

At time n, the investor decides the size of capital stock at time $n + 1$ (that is, X_{n+1}). The cost is given as

$$X_n c(X_{n+1}/X_n),$$

and the optimization problem is, given that $(X_0, Z_0) = (x, z)$,

$$\sup E \sum_{n=0}^{\infty} \beta^n \left[U(X_n, Z_n) - X_n c(X_{n+1}/X_n) \right]$$

Assume the following conditions:

(a) The function D is continuous, strictly decreasing in q, and strictly increasing in z, with $D(0, z) > 0$ and $\lim_{q\to\infty} D(q, z) = 0$ for all z.

(b) The function U is uniformly bounded; i.e., $\lim_{q\to\infty} U(q, z) \leq A$ for some constant A for all z.

(c) The function c is continuously differentiable and that for some $\delta \in (0, 1)$, $c(x) = 0$ for $x \in [0, 1 - \delta]$ and c is strictly increasing and strictly convex on $(1 - \delta, \infty)$.

(d) The Markov chain $\{Z_n\}$ takes values in a bounded, non-negative interval S. The transition probability function P satisfies the Feller property. Furthermore, for every non-decreasing function $f : S \to \mathbb{R}$, the function

$$Tf(z) \doteq \int_S f(z') P(dz'|z)$$

is also non-decreasing.

Show that,

(a) The value function $v(x, z)$ is the unique, bounded continuous solution to the DPE.

(b) The value function v is strictly increasing in both arguments and strictly concave in its first argument.

(c) The optimal policy is a (single-valued) continuous function, say $g(x, z)$.

(d) For each z, $v(\cdot, z)$ is continuously differentiable. For each z, the function $g(\cdot, z)$ is strictly increasing but with a slope strictly less than 1. For each x, $g(x, \cdot)$ is non-decreasing.

(*Hint*: the control set is non-compact. To overcome this, consider the control set $U(x) \doteq [(1-\delta')x, M]$ for $x \in (0, M]$ and $U(x) = [(1-\delta')x, x]$ for $x > M$. M is very large positive number and $\delta' \in (\delta, 1)$.)