

**AM 166: Homework # 6 (due Tuesday, April 17)**

1. Assume that a large number of red balls and white balls are mixed up in an urn. Let  $\theta$  be the proportion of balls that are red, and assume that  $\theta$  is unknown to you.
  - (a) Assign a uniform prior distribution on  $\theta$ . That is,  $\pi(\theta) = 1$  for all  $\theta \in [0, 1]$ . What is your prior probability that  $\theta$  is bigger than 0.5?
  - (b) Suppose that you randomly draw five times from the urn, and you get four red balls and one white ball. Identify your posterior distribution of  $\theta$  (up to a normalization constant).
  - (c) Compute the normalization constant in (b) so that you can identify the posterior distribution of  $\theta$  exactly.
  - (d) Plot the prior density and posterior density on the same graph (you may want to use STATA).
  - (e) Calculate the mode, mean, and standard deviation of the posterior distribution of  $\theta$ . What is your posterior probability that  $\theta$  is bigger than 0.5?
  
2. In a documentary film *A Private Universe*, only 2 out of 23 random samples of Harvard graduates were able to correctly answer the question: “Why is it hotter in summer than in winter?” Let  $\theta$  be the proportion of Harvard graduates who can answer the question correctly, and suppose that you assign  $\theta$  a uniform prior.
  - (a) Identify the posterior distribution of  $\theta$  exactly. Note that it is a Beta distribution.
  - (b) If you were to randomly select another Harvard graduate, what would be your probability that he/she will be able to answer the question correctly. [This is called Bayesian Prediction]. You may want to use the formula for the expected values of Beta distributions.
  
3. Setting parameters for a Beta prior distribution: suppose your prior distribution for  $\theta$ , the proportion of Californians who support the death penalty, is  $\text{Beta}(\alpha, \beta)$  with mean 0.6 and standard deviation 0.3.
  - (a) Determine the hyperparameters  $\alpha$  and  $\beta$  for the prior distribution.
  - (b) A random sample of 1000 Californians is taken, and 650 support the death penalty. What are your posterior mean and standard deviation for  $\theta$ ?
  
4. Normal distribution with unknown mean: a random sample of  $n$  students is drawn from a large population, and their weights are measured. The average weight of the  $n$  sampled students is  $\bar{y} = 150$  pounds. Assume the weights in the population are normally distributed with unknown mean  $\theta$  and standard deviation 20 pounds. Suppose your prior distribution for  $\theta$  is  $N(180, 40^2)$ .
  - (a) Identify your posterior distribution for  $\theta$  exactly. [Your answer will be a function on  $n$ ].
  - (b) For  $n = 10$  calculate your posterior probability that  $\theta$  is bigger than 180. Do the same thing for  $n = 100$ .

5. The length of life of a light bulb has an exponential distribution with unknown rate  $\theta$ . Suppose that the prior distribution of  $\theta$  is a Gamma distribution with coefficient of variation 0.5. (The *coefficient of variation* is defined as the standard deviation divided by the mean). A random sample of light bulbs is to be tested and the lifetime of each obtained. If one wants the coefficient of variation of the posterior distribution to be reduced to 0.1, how many light bulbs need to be tested?