

HW 8 Solution

7.23 CLT states that $Y_n = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$ converges in distribution to a standard normal random variable, Z .

Here, $n=100$, $\sigma=2.5$, and the approximation is

$$\begin{aligned} P(|\bar{X} - \mu| \leq 0.5) &= P(-.5 \leq \bar{X} - \mu \leq .5) \\ &= P\left(\frac{-.5 \times 10}{2.5} \leq Z \leq \frac{.5 \times 10}{2.5}\right) = P(-2 \leq Z \leq 2) \\ &= 1 - 2(1.0228) = .9544 \end{aligned}$$

7.37 Denote $X_i =$ the length of life for the i^{th} lamp, $i=1, 2, \dots, 25$.

Given X_i 's are independent with $\mu=50$ & $\sigma=4$.

By CLT, $Y_n = \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} = \frac{n(\bar{X} - \mu)}{\sqrt{n}\sigma} = \frac{\sum X_i - n\mu}{\sqrt{n}\sigma}$ converges to

a standard normal R.V. in distribution, then

$$P\left(\sum_{i=1}^{25} X_i \geq 1300\right) = P\left(\frac{\sum X_i - n\mu}{\sqrt{n}\sigma} \geq \frac{1300 - 1250}{5 \times 4}\right) = P(Z > 2.5) = .0062$$

7.47 Let $X = \#$ of persons not showing up for a given flight
 $X \sim$ Binomial (n, p), where $n=160$ & $p=.05$.

If there is to be a seat available for everyone planning to fly, then there must be at least 5 persons not showing up, i.e. $X \geq 5$.

Calculate $\mu = np = 160(.05) = 8$, $\sigma = \sqrt{npq} = 2.76$

A correction for continuity (Refer to example 7.10) is made to include the entire area under the rectangle associated with $X=5$, and the approximation becomes $P(X \geq 4.5)$.

Correspondingly, $Z = \frac{4.5 - 8}{2.76} = -1.27$

so that $P(X \geq 4.5) = P(Z \geq -1.27) = 1 - P(Z \leq -1.27) = 1 - .1020 = .8980$

8.8 a. $Y_i \sim \text{Uniform}(\theta, \theta+1) \Rightarrow EY_i = \theta + \frac{1}{2} \quad (i=1, 2, \dots, n)$
 $\Rightarrow E(\bar{Y}) = \theta + \frac{1}{2}$

& the bias is $B = E(\bar{Y}) - \theta = \frac{1}{2}$

b. An unbiased estimator of θ can be constructed by using $\hat{\theta} = \bar{Y} - \frac{1}{2}$, which has $E(\hat{\theta}) = \theta$.

c. If \bar{Y} is used as an estimator, then

$$V(\bar{Y}) = \frac{1}{n} V(Y) = \frac{1}{n} \int_0^{\theta+1} [y - (\theta + \frac{1}{2})]^2 dy = \frac{1}{12n}$$

& $\text{MSE} = V(\bar{Y}) + B^2$ (See Def'n 8.4) $= \frac{1}{12n} + \frac{1}{4}$

8.42(a) $\hat{p} = 268/500 = .536$

Therefore, an approximate 98% confidence interval for p is

$$\hat{p} \pm z_{0.01} \sqrt{\frac{\hat{p}\hat{q}}{n}} \quad (\text{see 2nd line on P391})$$

$$= .536 \pm 2.33 \sqrt{\frac{(.536)(.464)}{500}} = .536 \pm .052 = (.484, .588)$$

8.46 Given $n=75$, $\bar{y} = 4.2$, $s = 1.5$ & $\alpha = 0.05 \Rightarrow z_{0.025} = 1.96$

So, a 95% confidence interval is

$$\bar{y} \pm z_{\alpha/2} \left(\frac{s}{\sqrt{n}} \right) = 4.2 \pm 1.96 \left(\frac{1.5}{\sqrt{75}} \right) = 4.2 \pm .34 = (3.86, 4.54)$$

8.47 The 95% confidence interval is

Refer to example 8.8,

$$\rightarrow (\bar{y}_1 - \bar{y}_2) \pm z_{0.025} \sqrt{\left(\frac{s_1}{\sqrt{n_1}} \right)^2 + \left(\frac{s_2}{\sqrt{n_2}} \right)^2}$$

$$\begin{aligned} & \text{Var}(\bar{y}_1 - \bar{y}_2) \\ &= \text{Var}\bar{y}_1 + \text{Var}\bar{y}_2 \\ &= \frac{s_1^2}{n_1} + \frac{s_2^2}{n_2} \end{aligned}$$

$$= (1167.1 - 140.9) \pm 1.96 \sqrt{\frac{(24.3)^2}{30} + \frac{(17.6)^2}{30}}$$

$$= 26.2 \pm 10.74 = (15.46, 36.94)$$

8.53 The 98% confidence interval is (notice $z_{\alpha/2} = z_{0.01} = 2.33$)

$$(\hat{p}_1 - \hat{p}_2) \pm 2.33 \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$
$$= (0.18 - 0.12) \pm 2.33 \sqrt{\frac{(0.18)(0.82)}{100} + \frac{(0.12)(0.88)}{100}}$$

$$= 0.06 \pm 0.12 = (-0.06, 0.18)$$

B/C $(-0.06, 0.18)$ includes 0, there is not enough evidence here to suggest that one line produces a higher proportion of defectives than the other.

8.62 The 95% confidence interval for μ is

$$\hat{\mu} \pm z_{0.025} \left(\frac{s}{\sqrt{n}} \right)$$

$$\text{we know } z_{0.025} \left(\frac{s}{\sqrt{n}} \right) = 1.96 \left(\frac{0.5}{\sqrt{n}} \right) = 0.1 \Rightarrow n = 97$$

It's not valid if we select all our water specimens from a single rainfall since all the observations should be independent.

8.63 The parameter to be estimated is $\mu_1 - \mu_2$

Given $1 - \alpha = 0.9$, $z_{0.05} = 1.645$ & $\sigma_1^2 = \sigma_2^2 = 0.25$, $n_1 = n_2 = n$.

$$\text{We require } 1.645 \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \leq 0.1$$

$$1.645 \sqrt{\frac{0.25 + 0.25}{n}} \leq 0.1 \Rightarrow n \geq 135.3$$

So, $n = 136$ samples should be selected at each location.

9.2 a. Since $EY_i = \mu$, $i=1, 2, \dots, n$

$$\therefore E(\hat{\mu}_1) = \frac{1}{2} [E(Y_1) + E(Y_2)] = \mu$$

$$E(\hat{\mu}_2) = \frac{1}{4} E(Y_1) + \frac{1}{2(n-2)} \sum_{i=2}^{n-1} E(Y_i) + \frac{1}{4} E(Y_n)$$

$$= \left(\frac{1}{4} + \frac{1}{2(n-2)} \cdot (n-2) + \frac{1}{4} \right) \mu = \mu$$

$$E(\hat{\mu}_3) = E\bar{Y} = \frac{1}{n} \sum_{i=1}^n E(Y_i) = \frac{1}{n} \cdot n\mu = \mu$$

b. $V(\hat{\mu}_1) = \frac{1}{4} V(Y_1) + \frac{1}{4} V(Y_2) = \frac{1}{4} \times \sigma^2 \times 2 = \frac{1}{2} \sigma^2$

$$V(\hat{\mu}_2) = \frac{1}{4^2} V(Y_1) + \frac{1}{[2(n-2)]^2} \sum_{i=2}^{n-2} V(Y_i) + \frac{1}{4^2} V(Y_n)$$

$$= \left[\frac{1}{16} + \frac{1}{4(n-2)} + \frac{1}{16} \right] \sigma^2 = \frac{n}{8(n-2)} \sigma^2$$

$$V(\hat{\mu}_3) = \frac{1}{n} V(Y_i) = \frac{1}{n} \sigma^2$$

$$\therefore \text{eff}(\hat{\mu}_3, \hat{\mu}_1) = \frac{V(\hat{\mu}_1)}{V(\hat{\mu}_3)} = \frac{\frac{1}{2}}{\frac{1}{n}} = \frac{n}{2}$$

$$\text{eff}(\hat{\mu}_3, \hat{\mu}_2) = \frac{V(\hat{\mu}_2)}{V(\hat{\mu}_3)} = \frac{\frac{n}{8(n-2)}}{\frac{1}{n}} = \frac{n^2}{8(n-2)}$$

9.3 For $Y_i \sim \text{Uniform}(\theta, \theta+1)$, $E(Y_i) = \theta + \frac{1}{2}$, $V(Y_i) = \frac{1}{12}$.

$$E(\hat{\theta}_1) = E(\bar{Y} - \frac{1}{2}) = \theta + \frac{1}{2} - \frac{1}{2} = \theta$$

$$\& V(\hat{\theta}_1) = V(\bar{Y}) = \frac{1}{n} V(Y_i) = \frac{1}{12n}$$

Recall the density function of $Y_{(n)}$ (see Section 6.7) is

$$g_n(y) = n [F(y)]^{n-1} f(y), \text{ where } F(y) = \int_{\theta}^y dt = y - \theta$$

$$\& f(y) = 1$$

$$\therefore g_n(y) = n (y - \theta)^{n-1} \text{ for } \theta \leq y \leq \theta + 1$$

$$\therefore E(\hat{\theta}_2) = E(Y_{(n)}) - \frac{n}{n+1} = \int_{\theta}^{\theta+1} y \cdot n (y - \theta)^{n-1} dy - \left(\frac{n}{n+1} \right)$$

$$\stackrel{x=y-\theta}{=} \int_0^1 n(x+\theta) x^{n-1} dx - \left(\frac{n}{n+1} \right) = \left(\frac{n}{n+1} + \theta \right) - \frac{n}{n+1} = \theta$$

$$\& V(\hat{\theta}_2) = V(Y_{(n)}) = E(Y_{(n)}^2) - [E(Y_{(n)})]^2 = \int_{\theta}^{\theta+1} y^2 \cdot n (y - \theta)^{n-1} dy$$

$$- \left(\frac{n}{n+1} + \theta \right)^2 \stackrel{x=y-\theta}{=} \int_0^1 (x+\theta)^2 \cdot n x^{n-1} dx - \left(\frac{n}{n+1} + \theta \right)^2 = \frac{n}{(n+2)(n+1)^2}$$

$$\therefore \text{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{V(\hat{\theta}_2)}{V(\hat{\theta}_1)} = \frac{\frac{n}{(n+2)(n+1)^2}}{\frac{1}{12n}} = \frac{12n^2}{(n+2)(n+1)^2}$$