Chapters 3~4. A Few Topics

Moment Generating Functions

Given a random variable X, its moment generating function $M(t)$ is defined as

$$M(t) \doteq E\left[\exp\{tX\}\right].$$

Remark: It is possible that moment generating function takes infinite value for all $t \neq 0$. Say, Cauchy distribution with density

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \qquad x \in \mathbb{R}.$$

Assumption: We always assume moment generating function exists in the sense that there is a number $b > 0$ such that $M(t)$ is finite for all $|t| < b$.

Why the name?

Use expansion

$$\exp\{x\} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

We have

$$M(t) = E\left[1 + tX + \frac{t^2X^2}{2!} + \cdots + \frac{t^nX^n}{n!} + \cdots\right],$$

or

$$M(t) = 1 + tE[X] + \frac{t^2}{2!}E[X^2] + \cdots + \frac{t^n}{n!}E[X^n] + \cdots$$

Remark: $E[X^n]$ is called the n-th moment. Moreover,

$$\left.\frac{d^nM(t)}{dt^n}\right|_{t=0} = E[X^n]$$

Examples

- Discrete random variables: $q = 1 - p$.

Distribution	$M(t)$
Binomial $B(n; p)$	$\left[pe^t + q\right]^n$
Geometric with probability of success p	$\dfrac{pe^t}{1 - qe^t}$
Possion with parameter λ	$e^{\lambda(e^t - 1)}$

- Continuous random variables:

Distribution	$M(t)$
Uniform $[0, 1]$	$\dfrac{e^t - 1}{t}$
Exponential with rate λ	$\dfrac{\lambda}{\lambda - t}, \quad t < \lambda$
Normal $N(\mu, \sigma^2)$	$\exp\left\{\mu t + \dfrac{t^2\sigma^2}{2}\right\}$

Theorem: Moment generating function uniquely determines the distribution.

Proof. "Probability and Measure" by Patrick Billingsley, Theorem 22.2.

Examples

Determine the distribution.

1. $M(t) = [0.7e^t + 0.3]^4$.
2. $M(t) = \dfrac{2e^t}{5 - 3e^t}$.
3. $M(t) = \dfrac{1}{1 - 2t}$.
4. $M(t) = 0.2e^{-t} + 0.3e^{2t} + 0.5$.

Tchebysheff's Inequality

Let X be a random variable with mean μ and standard deviation σ. Then for any $k > 0$,

$$P\left(|X-\mu| \geq k\sigma\right) \leq \frac{1}{k^2}.$$

Proof. Let

$$h(x) \doteq (x-\mu)^2, \qquad g(x) \doteq k^2\sigma^2 \cdot 1_{\{|x-\mu|\geq k\sigma\}}$$

Then $h(x) \geq g(x)$. It follows that

$$E[h(X)] \geq E[g(X)],$$

or

$$\sigma^2 \geq k^2\sigma^2 P(|X-\mu| \geq k\sigma).$$

$\square$