

CHAPTERS 7. CENTRAL LIMIT THEOREM AND LAW OF LARGE
NUMBERS

Two most fundamental results in probability is [Central Limit Theorem \(CLT\)](#) and [Law of Large Numbers \(LLN\)](#)

Law of Large Numbers (LLN)

Suppose X_1, X_2, \dots is a sequence of independent identically distributed (iid) random variables with $E[X_i] = \mu$. Then

$$\lim_{n \rightarrow \infty} \frac{X_1 + X_2 + \dots + X_n}{n} = \mu$$

with probability one.

Simulation demonstration.

Central Limit Theorem (CLT)

Suppose X_1, X_2, \dots is a sequence of independent identically distributed (iid) random variables with $E[X_i] = \mu$ and $\text{Var}[X_i] = \sigma^2$. Then as $n \rightarrow \infty$,

$$\frac{X_1 + X_2 + \dots + X_n - n\mu}{\sqrt{n}\sigma} \Rightarrow N(0, 1).$$

The convergence “ \Rightarrow ” means

$$\lim_{n \rightarrow \infty} P \left(\frac{X_1 + X_2 + \dots + X_n - n\mu}{\sqrt{n}\sigma} \leq y \right) = \Phi(y)$$

for every $y \in \mathbb{R}$.

Remark: Formally speaking, CLT claims that the distribution of the “sample mean”

$$\bar{X} \doteq \frac{X_1 + X_2 + \cdots + X_n}{n}$$

is approximately

$$N\left(\mu, \frac{\sigma^2}{n}\right)$$

Simulation demonstration.

FORMAL ARGUMENT FOR LLN AND CLT

Use moment generating functions

NORMAL APPROXIMATION

1. Suppose the IQ for college students has mean 110, and standard deviation 20. A random sample of $n = 100$ students are selected. Approximate the probability that the sample IQ average score is below 108?

2. Normal approximation to Binomial distributions.

Suppose X is $B(n; p)$. Write X as

$$X = Y_1 + Y_2 + \cdots + Y_n$$

where $\{Y_1, Y_2, \dots, Y_n\}$ are iid Bernoulli random variables with

$$P(Y_i = 1) = p, \quad P(Y_i = 0) = q = 1 - p.$$

$E[Y_i] = p$, $\text{Var}[Y_i] = pq$. By CLT,

$$\frac{X}{n}$$

is approximately

$$N\left(p, \frac{pq}{n}\right)$$

Remark: The normal approximation works well when

$$n > 9 \cdot \frac{\max\{p, q\}}{\min\{p, q\}}$$

3. Example: Suppose X is $B(100; 0.3)$. Compute $P(X > 35)$.

Solution:

$$P(X > 35) = P\left(\frac{X}{100} > 0.35\right)$$

Let

$$Z = \frac{\frac{X}{100} - 0.3}{\sqrt{\frac{0.3 \cdot 0.7}{100}}}$$

Then

$$P(X > 35) \approx P\left(Z > \frac{0.35 - 0.3}{\sqrt{\frac{0.3 \cdot 0.7}{100}}}\right) = P(Z > 1.09) = 0.1379$$