Chapters 6. Functions of Random Variables

Question: Given a collection of random variables $X_1, X_2, \ldots, X_d$ and a function h, what is the distribution of the random variable

$$Y \doteq h(X_1, X_2, \ldots, X_d).$$

Solution: Several approaches, including

1. THE METHOD OF DISTRIBUTION FUNCTION:
   Compute the cdf for Y, i.e. $P(Y \leq y)$.
2. THE METHOD OF MOMENT GENERATING FUNCTIONS:
   Compute the moment generating function Y.

The method of distribution function

1. Suppose X is uniform on $(0, 1)$. What is the distribution of $-\ln(X)$?

2. Suppose X has cdf F and density f. Find the cdf and density for $aX + b$ where $a > 0$.

3. Let X and Y be indepedent exponential random variables with rates λ and μ respectively. What is the distribution of $\min\{X, Y\}$?

4. Let X and Y be Poisson distributions with parameters λ and μ respectively, what is the distribution of $X + Y$?

5. Suppose X and Y are independent, identically distributed exponential random variables with rate λ. What is the distribution of $X/(X+Y)$?

6. Suppose X is a continuous random variable with cdf F. What is the distribution of $F(X)$?

The method of moment generating functions

Moment generating function uniquely determines the distribution!

1. Suppose X is $N(\mu, \sigma^2)$. Find the distribution of $aX + b$.

2. Suppose X_1 is $N(\mu_1, \sigma_1^2)$ and X_2 is $N(\mu_2, \sigma_2^2)$. Assume X_1 and X_2 are independent, what is the distribution of $X_1 + X_2$?

Order Statistics

Let $X_1, X_2, \ldots, X_n$ be iid continuous random variables with common density $f(x)$. The order-statistics is denoted by $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$, where

$$X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$$

is the re-ordering of $\{X_1, X_2, \ldots, X_n\}$ in an increasing order.

1. What is the cdf for $X_{(1)}$? $X_{(n)}$?

2. Suppose $X_1, X_2, \ldots, X_n$ are iid uniform random variables on $[0, \theta]$. Find $E[X_{(n)}]$.

The joint density of order statistics

$$f_{(1)(2)\cdots(n)}(x_1, x_2, \ldots, x_n) = \begin{cases} n! f(x_1) f(x_2) \ldots f(x_n) & , \quad x_1 \leq x_2 \leq \cdots \leq x_n \\ 0 & , \quad \text{otherwise} \end{cases}$$