

CHAPTER 4. CONTINUOUS RANDOM VARIABLES

REVIEW

- **Continuous random variable:** A random variable that can take any value on an interval of \mathbb{R} .
- **Distribution:** A non-negative density function $f : \mathbb{R} \rightarrow \mathbb{R}_+$ such that

$$P(X \in I) = \int_I f(x) dx$$

for every subset $I \subset \mathbb{R}$, where

$$\int_{\mathbb{R}} f(x) dx = 1.$$

- **Graphic description:** (1) density f , (2) cdf F .
- **Relation between density f and cdf F :**

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(y) dy$$

$$f(x) = F'(x).$$

- Example: Suppose X has density f that takes the following form:

$$f(x) = \begin{cases} cx(2-x) & , \quad 0 \leq x \leq 2, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Determine c , cdf F , and find $P(X < 1)$.

Expected Value, Variance, and Standard Deviation

Continuous random variable X with density f .

- Expectation (expected value, mean, “ μ ”):

$$EX \doteq \int_{-\infty}^{\infty} x f(x) dx$$

THEOREM: Consider a function h and random variable $h(X)$. Then

$$E[h(X)] = \int_{-\infty}^{\infty} h(x)f(x) dx$$

Corollary: Let a, b be real numbers. Then $E[aX + b] = aEX + b$.

Corollary: Consider functions h_1, \dots, h_k . Then

$$E[h_1(X) + \dots + h_k(X)] = E[h_1(X)] + \dots + E[h_k(X)].$$

- Variance (“ σ^2 ”) and Standard deviation (“ σ ”):

$$\text{Var}[X] \doteq E[(X - EX)^2], \quad \text{Std}[X] \doteq \sqrt{\text{Var}X}.$$

Proposition: $\text{Var}[X] = 0$ if and only if $P(X = c) = 1$ for some constant c .

Proposition:

$$\text{Var}[X] = E[X^2] - (EX)^2.$$

Proposition: Let a, b be real numbers. Then $\text{Var}[aX + b] = a^2\text{Var}[X]$.

UNIFORM DISTRIBUTION

- Uniform distribution on $[0, 1]$. A random variable X with density

$$f(x) = \begin{cases} 1 & , \quad 0 \leq x \leq 1 \\ 0 & , \quad \text{otherwise.} \end{cases}$$

- Uniform distribution on $[a, b]$. A random variable X with density

$$f(x) = \begin{cases} \frac{1}{b-a} & , \quad a \leq x \leq b \\ 0 & , \quad \text{otherwise.} \end{cases}$$

- **Remark:** If X is uniformly distributed on $[0, 1]$, then $Y \doteq (b-a)X + a$ is uniformly distributed on $[a, b]$. (Find cdf, then density, for Y)
- **Remark:** A point is chosen at random from $[0, 1]$ means the observation is a random variable X with uniform distribution on $[0, 1]$.

EXAMPLE

Suppose customers arrive at post office randomly such that

1. The number of arrivals on any time interval with length s is Poisson distributed with parameter λs .
2. The numbers of arrivals on disjoint time intervals are independent.

Assume on time interval $[0, T]$, there is in total one customer. Let X denote the arrival time of this customer. Find the distribution of X .

EXPONENTIAL DISTRIBUTION

- Exponential distribution with rate λ . A random variable X with density

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , \quad 0 \leq x < \infty \\ 0 & , \quad \text{otherwise.} \end{cases}$$

$$EX = \frac{1}{\lambda}, \quad \text{Var}[X] = \frac{1}{\lambda^2}.$$

Remark: “Exponential distribution with rate λ ” is called in the textbook “Exponential distribution with parameter β ” where $\beta = 1/\lambda$.

MEMORYLESS PROPERTY

Suppose X is exponentially distributed with rate λ . Then

$$P(X > t + s | X > t) = P(X > s).$$

Comment: Exponential distribution is the **only** continuous distribution that has memoryless property.

EXAMPLE: The life length of a computer is exponentially distributed with mean 5 years. You bought an old (working) computer for 10 dollars. What is the probability that it would still work for more than 3 years?

NORMAL DISTRIBUTION

Normal distribution $N(\mu, \sigma^2)$. A random variable X is normally distributed with mean μ and variance σ^2 if it has density

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\}, \quad x \in \mathbb{R}.$$

1. f defines a probability density function.
2. $EX = \mu$.
3. $\text{Var}[X] = \sigma^2$.

STANDARD NORMAL DISTRIBUTION

The **standard normal** means $N(0, 1)$. Its cdf is often denoted by Φ , that is,

$$\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\} dy$$

- Suppose Z is $N(0, 1)$, then

$$P(Z > x) = P(Z < -x) = \Phi(-x).$$

In particular,

$$\Phi(x) + \Phi(-x) = 1.$$

STANDARDIZATION

Theorem: Suppose X has normal distribution $N(\mu, \sigma^2)$. Then its linear transform $Y \doteq aX + b$ is also normally distributed with distribution $N(a\mu + b, a^2\sigma^2)$.

Corollary: Suppose X has distribution $N(\mu, \sigma^2)$. Then its standardization

$$Z \doteq \frac{X - \mu}{\sigma}$$

is standard normal.

EXAMPLES

1. Suppose X has distribution $N(40, 100)$. Find $P(50 < X < 60)$.
2. Suppose X has distribution $N(\mu, \sigma^2)$. Then

$$P(|X - \mu| > k\sigma) = 2\Phi(-k).$$

In particular,

k		1	2	3
$2\Phi(-k)$		32%	5%	0.3%

REMARK: Normal distribution is often used as approximation of **non-negative** distributions. For example, SAT scores, IQ, GRE scores