Am165 HW9

9.34 9.37 9.44 9.51 9.55 9.72 9.74 9.79 9.81

9.34) $f(y|p) = p(1-p)^{y-1}$

$L(y_1, \cdots, y_n|p) = p^n(1-p)^{\Sigma y_i - n} = p^n(1-p)^{n\bar{y}-n}$

let $g(\bar{y}, p) = p^n(1-p)^{n\bar{y}-n}$ and $h(y_1, \cdots, y_n) = 1$

then by Thm 9.4, $\bar{Y}$ is sufficient for p.

9.37) The likelihood is $L(y_1, \cdots, y_n|\theta) = a(\theta)^n \prod b(y_i) e^{-c(\theta)\Sigma d(y_i)}$

let $g(\Sigma d(Y_i), \theta) = a(\theta)^n e^{-c(\theta)\Sigma d(y_i)}$ and $h(y_1, \cdots, y_n) = \prod b(y_i)$

Then by Thm 9.4 $\sum_{i=1}^{n} d(Y_i)$ is sufficient for θ.

9.44) Note that $f(y_1, y_2, \cdots, y_n|\theta) = \begin{cases} \prod_{i=1}^{n} 3y_i^2/\theta^3 & \text{for } 0 \le y_i \le \theta \\ 0 & \text{otherwise} \end{cases}$

Note also the condition that $0 \le y_i \le \theta$ for $i = 1, \cdots, n$ can be written as $0 \le y_{(1)}$ and $y_{(n)} \le \theta$ (if the minimum is greater than 0, then so must be the remainder of the y_i; the same argument applies for the maximum being less than θ) The likelihood can then be written as

$$L(y_1, \cdots, y_n|\theta) = \frac{3^n}{\theta^{3n}} \left(\prod_{i=1}^{n} y_i\right) I(0 \le y_{(1)}) I(y_{(n)} \le \theta)$$

Thus with $g(y_{(n)}, \theta) = \frac{3^n}{\theta^{3n}} I(y_{(n)} \geq \theta)$ and

$h(y_1, \cdots, y_n) = \left(\prod_{i=1}^{n} y_i\right) I(0 \le y_{(1)})$, we can see that by the factorization theorem, $Y_{(n)}$ is sufficient for θ.

9.51) with Y a poisson random variable with parameter λ, it is necessary to find the MVUE for

$$E(C) = 3E(Y^2) = 3(V(Y) + [E(Y)]^2) = 3(\lambda + \lambda^2)$$

In Exercise 9.31 it was determined that $\sum_{i=1}^{n} Y_i$ is sufficient for λ and thus for λ^2 and $3(\lambda+\lambda^2)$.

If a function of $\sum_{i=1}^{n} Y_i$ that is unbiased for $3(\lambda+\lambda^2)$ can be found, then this function will be MVUE. Note that as $\bar{Y}$ is distributed as poisson$(n\lambda)$ we can easily calculate

$$E(\bar{Y}^2) = V(\bar{Y}) + [E(\bar{Y})]^2 = \frac{\lambda}{n} + \lambda^2$$

and

$$E\left(\frac{\bar{Y}}{n}\right) = \frac{1}{n}E(\bar{Y}) = \frac{\lambda}{n}$$

Then we have $\lambda^2 = E(\bar{Y}^2) - E\left(\frac{\bar{Y}}{n}\right)$ and $\lambda = E(\bar{Y})$.

$E(C) = 3E\left[\bar{Y}^2 - \left(\frac{\bar{Y}}{n}\right) + \bar{Y}\right]$ and the MVUE is $3\left[\bar{Y}^2 + \bar{Y}\left(1 - \frac{1}{n}\right)\right]$.

9.55) a. First note that the distribution function corresponding to $f(y)$ is

$$F(y) = \begin{cases} 1 & \text{if } y > \theta \\ y^3/\theta^3 & \text{for } 0 \le y \le \theta \\ 0 & \text{for } y < 0 \end{cases}$$

Then the density of $Y_{(n)}$ is $f_{Y_{(n)}}(y) = n f(y)(F(y))^{n-1}$

$$= \begin{cases} n\,3\frac{y^2}{\theta^3}\left(\frac{y^3}{\theta^3}\right)^{n-1} = 3ny^{3n-1}/\theta^{3n} & \text{for } 0 \le y \le \theta \\ 0 & \text{otherwise} \end{cases}$$

b. We know that the UMVUE will be based on $Y_{(n)}$ as it is a complete sufficient statistic. All we need to do is properly scale it so that it is unbiased.
Note

$$E(Y_{(n)}) = \int_0^\theta 3ny^{3n}/\theta^{3n}\,dy = \frac{3ny^{3n+1}}{(3n+1)\theta^{3n}}\Big|_0^\theta = \frac{3n}{3n+1}\theta$$

Thus $\frac{3n+1}{3n} Y_{(n)}$ is unbiased for θ. It is then UMVUE as $Y_{(n)}$ is complete sufficient.

9.72) a. The likelihood function is

$$L = \prod_{i=1}^{n} \frac{\lambda^{y_i} e^{-\lambda}}{y!} = \frac{\lambda^{\Sigma y_i} e^{-n\lambda}}{\prod_{i=1}^{n} y_i!}$$

and $\ln L = (\Sigma y_i)\ln\lambda - n\lambda - \Sigma \ln y_i!$
So that $(\frac{d}{d\lambda})[\ln L] = (\Sigma \frac{y_i}{\lambda}) - n$. Equating the derivative to 0, we obtain $\frac{\Sigma y_i}{\hat\lambda} - n = 0$

or $\hat\lambda = \frac{\Sigma Y_i}{n} = \bar{Y}$.

b. Recalling that $E(Y_i) = \lambda$ and $V(Y_i) = \lambda$, we obtain

$$E(\hat\lambda) = \frac{\sum_{i=1}^{n} E(Y_i)}{n} = \lambda$$

and

$$V(\hat\lambda) = \frac{\sum_{i=1}^{n} V(Y_i)}{n^2} = \frac{\lambda}{n}$$

c. Since $E(Y_i) = \lambda$ and $V(Y_i) = \lambda < \infty$, the law of large numbers applies and we conclude that $\hat{\lambda}$ converges in probability to λ. Hence $\hat{\lambda}$ is consistent for λ.

d. The MLE of λ was found in part a. to be $\hat{\lambda} = \bar{Y}$. Then, the MLE for $e^{-\lambda}$ is $e^{-\bar{Y}}$.

9.74) a. The likelihood function is

$$L = \prod_{i=1}^{n} \frac{r}{\theta} y_i^{r-1} e^{-y_i^r/\theta} = \frac{r^n}{\theta^n} \prod_{i=1}^{n} y^{r-1} e^{-\Sigma y_i^r/\theta}$$

$$= g(u, \theta) h(y_1, y_2, \cdots, y_n)$$

where

$$u = \sum_{i=1}^{n} y_i^r \qquad g(u,\theta) = \frac{r^n}{\theta^n} e^{-u/\theta}, \quad h(y_1, \cdots, y_n) = \prod_{i=1}^{n} y_i^{r-1}$$

Hence ΣY_i^r is a sufficient statistic for θ.

b. Consider $\ln L = n \ln r - n \ln \theta + (r-1) \sum_{i=1}^{n} \ln y_i - \sum_{i=1}^{n} \frac{y_i^r}{\theta}$

and $\frac{d}{d\theta} \ln L = \frac{-n}{\theta} + \frac{\Sigma y_i^r}{\theta^2}$

Equating the derivative to 0, the estimator is obtained.

$$\frac{-n}{\hat{\theta}} + \frac{\Sigma y_i^r}{\hat{\theta}^2} = 0 \qquad \text{or} \qquad -n\hat{\theta} + \Sigma y_i^r = 0$$

or $\hat{\theta} = \frac{\Sigma Y_i^r}{n}$.

c. The estimator $\hat{\theta}$ given in part b. is a function of the sufficient statistic. If it is unbiased, or could be adjusted to be unbiased, the MVUE of θ will be obtained.

Consider

$$E(Y_i^r) = \int_0^\infty \frac{r}{\theta} y^{2r-1} e^{-y^r/\theta}\, dy$$

let $x = y^r$; $dx = r y^{r-1} dy$, so that $E(Y_i^r) = \int_0^\infty \left(\frac{x}{\theta}\right) e^{-x/\theta} dx$
$= E(X)$

where X ~~was~~ has a gamma distribution with $\alpha = 1$, $\beta = \theta$

Then $E(Y_i^r) = \theta$ and

$$E(\hat{\theta}) = \frac{\sum_{i=1}^{n} E(Y_i^r)}{n} = \theta$$

Since $\hat{\theta}$ is unbiased for θ, it is the MVUE for θ.

9.79) Let p_1, p_2, p_3 be the proportions of voters in the population favoring candidates A, B, and C, respectively. Further, define the random variables n_1, n_2 and n_3 as the number of voters in a random sample of size n who favor candidates A, B, and C, respectively.

Note that

$$\sum_{i=1}^{3} p_i = 1 \quad \text{and} \quad \sum_{i=1}^{3} n_i = n$$

So that we may write $p_3 = 1 - p_1 - p_2$ and $n_3 = n - n_1 - n_2$, the random variables n_1, n_2, n_3 follow a multinomial probability distribution (see section 5.9 of the book), and the likelihood function is

$$L = \frac{n!}{n_1! n_2! n_3!} p_1^{n_1} p_2^{n_2} (1-p_1-p_2)^{n_3}$$

so that $\ln L = \ln K + n_1 \ln p_1 + n_2 \ln p_2 + n_3 \ln(1-p_1-p_2)$

Differentiating with respect to p_1 and p_2, we have

$$\frac{d\ln L}{dp_1} = \frac{n_1}{p_1} - \frac{n_3}{1-p_1-p_2} \quad \text{and} \quad \frac{d\ln L}{dp_2} = \frac{n_2}{p_2} - \frac{n_3}{1-p_1-p_2}$$

Set these two equations equal to 0 and solve simultaneously for $\hat{p}_1$ and $\hat{p}_2$.

(*) $n_1(1-\hat{p}_1-\hat{p}_2) - n_3\hat{p}_1 = 0$

$n_2(1-\hat{p}_1-\hat{p}_2) - n_3\hat{p}_2 = 0$

Adding the two equations, we have

$(n_1+n_2)(1-\hat{p}_1-\hat{p}_2) = (\hat{p}_1+\hat{p}_2)n_3$ or $\hat{p}_1+\hat{p}_2 = \frac{n_1+n_2}{n}$

$\Rightarrow$

$\hat{p}_1 = \frac{n_1}{n}$, $\hat{p}_2 = \frac{n_2}{n}$ and thus $\hat{p}_3 = \frac{n_3}{n}$

for the data given, $\hat{p}_1 = 0.30$ $\hat{p}_2 = 0.38$, $\hat{p}_3 = 0.32$

9.81). $P(Y=y) = \binom{2}{y} p^y (1-p)^{2-y}$ Our estimator $\hat{p}$ must be either 1/4 or 3/4. We choose based on which has the larger likelihood value given the data, Y. It is important to remember in this problem that the likelihood is a function of the parameter p. Therefore we have three possible likelihood functions depending, one for each value of the data, Y.

$L(0,p) = P(Y=0) = (1-p)^2$ implying $\hat{p} = \frac{1}{4}$ as

$L(0,\frac{1}{4}) = (1-\frac{1}{4})^2 > (1-\frac{3}{4})^2 = L(0,\frac{3}{4})$

$L(1,p) = P(Y=1) = 2p(1-p)$ implying $\hat{p}$ can be either $\frac{1}{4}$ or $\frac{3}{4}$ as

$$L(1,\tfrac{1}{4}) = 2 * \tfrac{1}{4}(1-\tfrac{1}{4}) = 2 * \tfrac{3}{4}(1-\tfrac{3}{4}) = L(1,\tfrac{3}{4})$$

$L(2,p) = P(Y=2) = p^2$ implying

$\hat{p} = \frac{3}{4}$ as $(\frac{1}{4})^2 < (\frac{3}{4})^2$

Notice the case when $Y=1$ is an instance where the maximum likelihood estimator is not a single unique value!

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