

5.20 a) The marginal distribution for Y_1 is $P(Y_1=0)=0.76$ and $P(Y_1=1)=0.24$; and the marginal distribution for Y_2 is given by $P(Y_2=0)=0.55$, $P(Y_2=1)=0.16$, and $P(Y_2=2)=0.29$;

$$b) P(Y_2=0 | Y_1=0) = \frac{0.38}{0.76} = 0.5; \quad P(Y_2=1 | Y_1=0) = \frac{0.14}{0.76} = 0.18$$

$$P(Y_2=2 | Y_1=0) = \frac{0.24}{0.76} = 0.32;$$

$$c) P(Y_1=0 | Y_2=2) = \frac{0.24}{0.29} = 0.83.$$

$$5.23 \quad a) f_1(y_1) = \int_{y_1}^1 6(1-y_2) dy_2 = 6(1-y_1) - 3 + 3y_1^2 = 3(1-y_1)^2, \quad \text{for } 0 \leq y_1 \leq 1$$

$$f_2(y_2) = \int_0^{y_2} 6(1-y_2) dy_1 = 6y_2(1-y_2) \quad \text{for } 0 \leq y_2 \leq 1.$$

$$b) P(Y_2 \leq 1/2 | Y_1 \leq 3/4) = \frac{F(3/4, 1/2)}{\int_0^{3/4} f_1(y_1) dy_1} = \frac{\int_0^{1/2} \int_0^{y_2} 6(1-y_2) dy_1 dy_2}{\int_0^{3/4} 3(1-y_1)^2 dy_1}$$

$$= \frac{1/2}{63/64} = \frac{32}{63};$$

$$c) f(y_1 | y_2) = \frac{f(y_1, y_2)}{f_2(y_2)} = \frac{6(1-y_2)}{6y_2(1-y_2)} = \frac{1}{y_2} \quad \text{if } 0 < y_2 < 1$$

for $0 \leq y_1 \leq y_2$;

$$d) f(y_2 | y_1) = \frac{f(y_1, y_2)}{f_1(y_1)} = \frac{6(1-y_2)}{3(1-y_1)^2} = \frac{2(1-y_2)}{(1-y_1)^2} \quad \text{if } 0 \leq y_1 < 1$$

for $y_1 \leq y_2 \leq 1$;

$$e) P(Y_2 \geq 3/4 | Y_1 = 1/2) = \int_{3/4}^1 f(y_2 | y_1 = 1/2) dy_2$$

$$= \int_{3/4}^1 8(1-y_2) dy_2 = 8 \times \frac{1}{4} - 4 \times \frac{7}{16} = \frac{1}{4}.$$

5.32 a) By definition $f(y_1, y_2) = f_1(y_1) f(y_2 | y_1)$

now $f_1(y_1) = 1$ for $0 \leq y_1 \leq 1$ where $Y_1 =$ amount stocked,
and $f(y_2 | y_1) = \frac{1}{y_1}$ for $0 \leq y_2 \leq y_1$ where $Y_2 =$ amount sold.

$$\Rightarrow f(y_1, y_2) = \begin{cases} \frac{1}{y_1} & \text{for } 0 \leq y_2 \leq y_1 \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$b) P(Y_2 > \frac{1}{4} | Y_1 = \frac{1}{2}) = \int_{\frac{1}{4}}^{\frac{1}{2}} f(y_2 | Y_1 = \frac{1}{2}) dy_2 = 2 \times \frac{1}{4} = \frac{1}{2};$$

$$c) P(Y_1 > \frac{1}{2} | Y_2 = \frac{1}{4}) = \int_{\frac{1}{2}}^1 f(y_1 | Y_2 = \frac{1}{4}) dy_1$$

$$f(y_1 | y_2) = \frac{f(y_1, y_2)}{f_2(y_2)} \quad \text{and} \quad f_2(y_2) = \int_{y_2}^1 \frac{1}{y_1} dy_1 = -\ln y_2$$

for $0 \leq y_2 \leq 1$

$$\text{so } P(Y_1 > \frac{1}{2} | Y_2 = \frac{1}{4}) = \int_{\frac{1}{2}}^1 \frac{1}{\ln 4} \cdot \frac{1}{y_1} dy_1 = \frac{\ln 2}{\ln 4} = \frac{1}{2}.$$

5.45 Dependent! Because the range of y_1 values on which $f(y_1, y_2)$ is defined is dependent on y_2 . (and vice versa)

5.53 Independent! Because $f(y_1, y_2)$ can be factored.

5.54 The desired probability is

$$P(X=1, Y=1) + P(X=2, Y=2) + \dots + \dots \\ = \sum_{n=0}^{\infty} (e^{-n} p) \cdot (e^{-n} p) = p^2 \cdot \sum_{n=0}^{\infty} (e^{-2})^n = \frac{p^2}{1 - e^{-2}}.$$

$$5.59 \quad a) f(y_1, y_2) = \begin{cases} \frac{1}{9} e^{-(y_1 + y_2)/3} & y_1 > 0 \quad y_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\begin{aligned}
 b) P(Y_1 + Y_2 \leq 1) &= \int_0^1 dy_2 \int_0^{1-y_2} \frac{1}{9} e^{-(y_1+y_2)/3} dy_1 \\
 &= \int_0^1 \frac{1}{3} e^{-\frac{y_2}{3}} \int_0^{1-y_2} \frac{1}{3} e^{-\frac{y_1}{3}} dy_1 \\
 &= \int_0^1 \frac{1}{3} e^{-\frac{y_2}{3}} \cdot (1 - e^{-\frac{(1-y_2)}{3}}) dy_2 \\
 &= -\frac{1}{3} e^{-\frac{1}{3}} + 1 - e^{-\frac{1}{3}} = 1 - \frac{4}{3} e^{-\frac{1}{3}}
 \end{aligned}$$

5-60 Y_1 denotes the bus arrival time and Y_2 the passenger arrival time, the joint density of Y_1 and Y_2 is

$$f(y_1, y_2) = f_1(y_1) f_2(y_2) = \begin{cases} 1 & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
 P(Y_2 \leq Y_1 \leq Y_2 + \frac{1}{4}) &= \int_0^{\frac{3}{4}} dy_2 \int_{y_2}^{y_2 + \frac{1}{4}} f(y_1, y_2) dy_1 + \\
 \int_{\frac{1}{4}}^1 dy_1 \int_{y_1 - \frac{1}{4}}^{y_1} f(y_1, y_2) dy_2 &= \frac{1}{32} + \frac{3}{16} = \frac{7}{32}
 \end{aligned}$$

5-68 $E(Y_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y_1 f(y_1, y_2) dy_1 dy_2 = \int_0^1 \int_0^1 y_1 (y_1 + y_2) dy_1 dy_2$

$$= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

by symmetry, $E(Y_2) = \frac{7}{12}$, so $E(30Y_1 + 25Y_2) = (30+25) \times \frac{7}{12} = 32.08$
($\frac{385}{12}$)

5-69 $f(y_1, y_2) = f_1(y_1) \cdot f_2(y_2)$, $f_1(y_1) = \frac{1}{4} y_1 e^{-y_1/2}$ and $f_2(y_2) = \frac{1}{2} e^{-y_2/2}$,

$$\begin{aligned}
 E\left(\frac{Y_2}{Y_1}\right) &= \int_0^{\infty} \int_0^{\infty} \frac{y_2}{y_1} f_1(y_1) \cdot f_2(y_2) dy_1 dy_2 = \int_0^{\infty} f_1(y_1) \cdot \frac{1}{y_1} dy_1 \cdot \int_0^{\infty} y_2 f_2(y_2) dy_2 \\
 &= E\left(\frac{1}{Y_1}\right) \cdot E(Y_2) = \int_0^{\infty} \frac{1}{4} e^{-\frac{y_1}{2}} dy_1 \cdot \int_0^{\infty} \frac{1}{2} y_2 \cdot e^{-\frac{y_2}{2}} dy_2 \\
 &= \left[-\frac{1}{2} e^{-y_1/2}\right]_0^{\infty} \cdot \left[1 - y_2 e^{-\frac{y_2}{2}}\right]_0^{\infty} + \left(-2 e^{-\frac{y_2}{2}}\right)_0^{\infty} \\
 &= \frac{1}{2} \cdot [0 + 2] = 1
 \end{aligned}$$