Sensitivity Analysis

Sensitivity analysis is concerned with how changes in an LP's parameters affect the optimal solution. It is important for several reasons. If a parameter changes, sensitivity analysis can often make it unnecessary to solve the problem again. Furthermore, a knowledge of sensitivity analysis often enables the analyst to determine from the original solution how changes in an LP's parameter changes its optimal solution.

From previous chapters, we know that *a simplex tableau (for a maximization problem) for a set of basic variables* $\mathbf{x_B}$ *is optimal if and only if the RHS in every row except Row (0) is non-negative and all the coefficients in Row (0) are non-negative.*

For the sake of completeness, we recall the following tableaus:

1. For canonical LP: Maximize $Z = c^T x$ such that $Ax = b$, $x \geq 0$.

Basic Variable	Row	Z	x	RHS
Z	(0)	1	$\mathbf{c_B}^T\mathbf{B}^{-1}A - c^T$	$\mathbf{c_B}^T\mathbf{B}^{-1}b$
$\mathbf{x_B}$	$\vdots$	$\mathbf{0}$	$\mathbf{B}^{-1}A$	$\mathbf{B}^{-1}b$

2. For LP in standard form: Maximize $Z = c^T x$ such that $Ax \leq b$, $x \geq 0$. Assume $b \geq 0$.

Basic Variable	Row	Z	x	$\mathbf{s}$	RHS
Z	(0)	1	$\mathbf{c_B}^T\mathbf{B}^{-1}A - c^T$	$\mathbf{c_B}^T\mathbf{B}^{-1}$	$\mathbf{c_B}^T\mathbf{B}^{-1}b$
$\mathbf{x_B}$	$\vdots$	$\mathbf{0}$	$\mathbf{B}^{-1}A$	$\mathbf{B}^{-1}$	$\mathbf{B}^{-1}b$

1 Changing the right-hand side of a constraint $b \to \bar{b}$:

It is easy to see that this will not change the coefficients in Row (0), but will change the RHS for all rows from $\mathbf{B}^{-1}b$ to $\mathbf{B}^{-1}\bar{b}$. *As long as the RHS for every row, except Row (0), remains non-negative, the current set of basic variables remains optimal. Otherwise, the set of basic variable is infeasible and therefore no longer optimal.*

2 Changing $c_j \to \bar{c}_j$ or $A_j \to \bar{A}_j$ when x_j is non-basic

In this case, only the column of this non-basic variable x_j in the simplex tableau will change. Indeed, the coefficient of x_j in Row (0) will change from $\mathbf{c_B}^T\mathbf{B}^{-1}A_j - c_j$ to $\mathbf{c_B}^T\mathbf{B}^{-1}\bar{A}_j - \bar{c}_j$, and the column of x_j, except Row (0), will become $\mathbf{B}^{-1}\bar{A}_j$. *As long as the coefficient of the non-basic variable x_j in Row (0) remains non-negative, the current set of basic variables remains optimal.* If this is the case, the optimal value and optimal solution remain *unchanged.* It also indicates that the coefficients of a non-basic variable are "insensitive" parameters.

When the coefficient of the non-basic variable x_j in Row (0) becomes negative, the current set of basic variables is still **feasible** (i.e. it yields a BFS) since the RHS of every row, except row (0), remains unchanged and non-negative. Therefore, we have an sub-optimal simplex tableau. In order to obtain the new optimal solution, simply perform the simplex algorithm.

Example: Consider a maximization LP

$$Z = \cdots + 2x_3 + \cdots$$

such that $Ax = b$, $x \geq 0$. Suppose in the optimal tableau, the variable x_3 is non-basic, and has coefficient 10 in Row (0). Assume now the coefficient for x_3 in the objective function is $2 + \triangle$ for some $\triangle$ which we are unsure of. For what range of $\triangle$ the current set of basic variable remains optimal?

Solution: In this case, the only change in the simplex tableau is the coefficient of x_3 in Row (0). However, coefficients in Row (0) is the vector $\mathbf{c_B}^T\mathbf{B}^{-1}A - c^T$. Since c^T now changes from $(\cdots, 2, \cdots)$ to $(\cdots, 2 + \triangle, \cdots)$. The coefficient of x_3 in Row (0) will become $10 - \triangle$. The range of $\triangle$ for the current basic variables to stay optimal is $10 - \triangle \geq 0$, or $\triangle \leq 10$. □

3 Changing $c_j \to \bar{c}_j$ when x_j is basic

In this case, $\mathbf{B}^{-1}$ and b remain unaffected. Hence the current set of basic variables is still feasible. However, since $\mathbf{c_B}$ changes, $\mathbf{c_B}^T\mathbf{B}^{-1}A$ will change. It is not difficult to see that a change in $\mathbf{c_B}$ may change more than one coefficient in Row (0). *As long as every coefficient in Row (0) remains non-negative, the current set of basic variable is still optimal.* Otherwise, it becomes sub-optimal.

If the current set of basic variables remains optimal, the optimal solution remains unchanged. However, the optimal value could change.

If the current set of basic variables becomes sub-optimal, simply perform simplex algorithm with the new Row (0).

4 Adding a new activity x_{n+1}, and adding c_{n+1} and A_{n+1}

In many situations, some new opportunities arise and we want to see if undertake these new opportunities will result in improvement. For example, suppose the original LP is to maximize $Z = c^T x$, such that $Ax \leq b$ and $x \geq 0$. Think of this LP as a production problem and we want to add a new product $(n+1)$. The amount of this new product will be x_{n+1} and its profit will be c_{n+1}. The resources used to produce a unit product $(n+1)$ is A_{n+1}. The new LP is therefore,

Maximize $\bar{Z} = c^T x + c_{n+1}x_{n+1}$ such that $Ax + A_{n+1}x_{n+1} \leq b$, $x = (x_1, x_2, \cdots, x_n)^T \geq 0$, $x_{n+1} \geq 0$.

It is not difficult to see that adding a new activity is just adding a new column in the simplex tableau corresponding to the new activity (in this case, x_{n+1}). The RHS for all the rows will remain the same. *As long as the coefficient in Row (0) for the new variable is non-negative, the current set of basic variable is still optimal.*

If the coefficient in Row (0) for the new variable becomes negative, the current set of basic variables is no longer optimal. In order to find the new optimal solution, use simplex algorithm, starting from the new tableau obtained.

Examples: Let us revisit the following LP:

$$\text{Maximize} \qquad Z = -x_1 + x_2$$

$$\begin{aligned} \text{such that} \qquad 2x_1 + x_2 &\leq 4 \\ x_1 + x_2 &\leq 2 \\ x_1, x_2 &\geq 0. \end{aligned}$$

The optimal tableau for this LP is

Basic Variable	Row	Z	x_1	x_2	s_1	s_2	RHS	Ratios
Z	(0)	1	2	0	0	1	2	
s_1	(1)	0	1	0	1	-1	2	
x_2	(2)	0	1	1	0	1	2	

We want to add a new variable x_3 and

$$c_3 = 1, \quad A_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

In other words, the LP now becomes

$$\text{Maximize} \qquad Z = -x_1 + x_2 + x_3$$

$$\begin{aligned} \text{such that} \qquad 2x_1 + x_2 + x_3 &\leq 4 \\ x_1 + x_2 - x_3 &\leq 2 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Will the current set of basic variables $\mathbf{x_B} = [s_1, x_2]$ remain optimal? If not, find the optimal solution.

Solution: In this case,

$$\mathbf{x_B} = [s_1, x_2], \qquad \mathbf{B}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{c_B}^T B^{-1} = [0 \ \ 1]$$

The new column in the simplex tableau corresponding to x_3 will have coefficient

$$\mathbf{c_B}^T \mathbf{B}^{-1} A_3 - c_3 = [0 \ \ 1] \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} - 1 = -2.$$

Therefore $\mathbf{x_B}$ is no longer optimal. Note

$$\mathbf{B}^{-1}A_3 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

we have the simplex tableau

Basic Variable	Row	Z	x_1	x_2	x_3	s_1	s_2	RHS	Ratios
Z	(0)	1	2	0	-2	0	1	2	
s_1	(1)	0	1	0	2$*$	1	-1	2	$2/2 = 1 \leftarrow$min
x_2	(2)	0	1	1	-1	0	1	2	

Basic Variable	Row	Z	x_1	x_2	x_3	s_1	s_2	RHS	Ratios
Z	(0)	1	3	0	0	1	0	4	
x_3	(1)	0	0.5	0	1	0.5	-0.5	1	
x_2	(2)	0	1.5	1	0	0.5	0.5	3	

Therefore, the new optimal solution is $Z^* = 4$ with $(x_1^*, x_2^*, x_3^*) = (0, 3, 1)$, $(s_1^*, s_2^*) = (0, 0)$. $\square$

5 Example

Suppose a company can manufacture three types of candy bars. Each candy bar consists of sugar and chocolate. Below is the composition and profit for each type of candy bar. 50 oz of sugar and 100 oz of chocolate are available.

Candy Bar	Sugar (ounce)	Chocolate (ounce)	Profit (cent)
A	1	2	3
B	1	3	7
C	1	1	5

The LP for maximizing the profit is as follows.

$$\text{Maximize} \qquad Z = 3x_1 + 7x_2 + 5x_3$$

such that

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & \leq & 50 \\ 2x_1 & + & 3x_2 & + & x_3 & \leq & 100 \end{array}$$

and $x_1, x_2, x_3 \geq 0$.

Suppose the optimal simplex tableau is

Basic Variable	Row	Z	x_1	x_2	x_3	s_1	s_2	RHS	Ratios
Z	(0)	1	3	0	0	4	1	300	
x_3	(1)	0	0.5	0	1	1.5	-0.5	25	
x_2	(2)	0	0.5	1	0	-0.5	0.5	25	

1. For what value of candy bar A profit does the current set of basic variables remain optimal? If the profit for candy bar A were 7 cents, what would be the new optimal solution?

2. For what value of candy bar B profit would the current set of basic variables remain optimal? If the profit for candy bar B were 13 cents, what would be the new optimal solution?

3. What is the most the company should pay for an additional ounce of sugar?

4. For what amount of available sugar would the current basic variables remain optimal?

5. If 60 0z of sugar is available, what would be the profit? How many of each candy bar should they make? What about if only 30 oz of sugar is available?

6. Suppose a type A candy bar used only 0.5 oz of sugar and 0.5 oz of chocolate. Should the company now make type A candy bars?

7. The company is considering a type D candy bar, which earns 17 cents profit and requires 3 oz of sugar and 4 oz of chocolate. Should the company manufacture any type D candy bar?

Solution: In the optimal tableau $\mathbf{x_B} = [x_3, x_2]$ and

$$\mathbf{B}^{-1} = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}, \quad \mathbf{c_B}^T B^{-1} = [4 \;\; 1].$$

1. Suppose the profit for the candy bar A is now $\bar{c}_1$. Note x_1 is a non-basic variable in the optimal tableau. Therefore, the change of c_1 only changes the coefficient of x_1 is Row (0), which now becomes

$$\mathbf{c_B}^T\mathbf{B}^{-1}A_1 - \bar{c}_1 = \mathbf{c_B}^T\mathbf{B}^{-1}A_1 - c_1 - (\bar{c}_1 - c_1) = 3 - (\bar{c}_1 - 3) = 6 - \bar{c}_1.$$

As long as the profit of candy bar A $\bar{c}_1 \leq 6$, the current set of basic variables remain optimal, and the optimal profit stays at 300 cents.

If $\bar{c}_1 = 7$, the current set of basic variables is no longer optimal, and we obtain a new simplex tableau

Basic Variable	Row	Z	x_1	x_2	x_3	s_1	s_2	RHS	Ratios
Z	(0)	1	-1	0	0	4	1	300	
x_3	(1)	0	0.5*	0	1	1.5	-0.5	25	$25/0.5 = 50 \leftarrow$min
x_2	(2)	0	0.5	1	0	-0.5	0.5	25	$25/0.5 = 50$

Basic Variable	Row	Z	x_1	x_2	x_3	s_1	s_2	RHS	Ratios
Z	(0)	1	0	0	2	7	0	350	
x_1	(1)	0	1*	0	2	3	-1	50	
x_2	(2)	0	0	1	-1	-2	1	0	

The new optimal solution is $Z^* = 350$, $(x_1^*, x_2^*, x_3^*) = (50, 0, 0)$ and $(s_1^*, s_2^*) = (0, 0)$.

2. Suppose now the profit for candy bar B is $\bar{c}_2$. We have

$$\mathbf{c_B} = [5 \;\; 7]^T \quad \rightarrow \quad \bar{\mathbf{c_B}} = [5 \;\; \bar{c}_2]^T.$$

Therefore Row (0) of the simplex tableau will become

$$\begin{aligned} \bar{\mathbf{c_B}}^T \mathbf{B}^{-1} A - \bar{c}^T &= [5 \;\; \bar{c}_2] \cdot \begin{bmatrix} 0.5 & 0 & 1 & 1.5 & -0.5 \\ 0.5 & 1 & 0 & -0.5 & 0.5 \end{bmatrix} - [3 \;\; \bar{c}_2 \;\; 5 \;\; 0 \;\; 0] \\ &= [0.5\bar{c}_2 - 0.5 \;\; 0 \;\; 0 \;\; 7.5 - 0.5\bar{c}_2 \;\; 0.5\bar{c}_2 - 2.5] \end{aligned}$$

The current set of basic variables remains optimal if and only if each coefficient in Row (0) is non-negative, or equivalently

$$0.5\bar{c}_2 - 0.5 \geq 0, \quad 7.5 - 0.5\bar{c}_2 \geq 0, \quad 0.5\bar{c}_2 - 2.5 \geq 0$$

or

$$5 \leq \bar{c}_2 \leq 15.$$

In case $\bar{c}_2 = 13$, $\mathbf{x_B} = [x_3, x_2]$ remains optimal, and the optimal solution x^* remains optimal, except the optimal value is now

$$Z^* = \bar{\mathbf{c_B}}^T \mathbf{B}^{-1} b = [5 \;\; 13] \cdot \begin{bmatrix} 25 \\ 25 \end{bmatrix} = 450.$$

3. The corresponding dual optimal solution is $y^* = [4 \;\; 1]^T$, which can also be regarded as the shadow price. Therefore, the company wants to pay at most 4 cent for an additional ounce of sugar, and at most 1 cent for an additional ounce of the chocolate.

4. Suppose now the available sugar is $\bar{b}_1$. The RHS for the rows, except Row (0), is now

$$\mathbf{B}^{-1}\bar{b} = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} \bar{b}_1 \\ 100 \end{bmatrix} = \begin{bmatrix} 1.5\bar{b}_1 - 50 \\ 50 - 0.5\bar{c}_1 \end{bmatrix}$$

Everything else remains the same. Therefore, the current set of basic variables $\mathbf{x_B}$ remains optimal as long as $\mathbf{B}^{-1}\bar{b} \geq 0$, or equivalently $100/3 \leq \bar{b}_1 \leq 100$.

5. If $\bar{b}_1 = 60$, the current set of basic variables $\mathbf{x_B}$ remains optimal. We have

$$\mathbf{B}^{-1}\bar{b} = \begin{bmatrix} 40 \\ 20 \end{bmatrix}, \quad \mathbf{c_B}^T \mathbf{B}^{-1}\bar{b} = [5 \;\; 7] \cdot \begin{bmatrix} 40 \\ 20 \end{bmatrix} = 340.$$

Hence, the optimal solution is

$$Z^* = 340, \quad (x_1^*, x_2^*, x_3^*) = (0, 20, 40), \quad (s_1^*, s_2^*) = (0, 0).$$

If $\bar{b}_1 = 30$, the current set of basic variables $\mathbf{x_B}$ is no longer feasible, and problem must be solved again.

6. First observe that x_1 is a non-basic variables in the optimal tableau. We have changed the coefficients of x_1 in the constraints to

$$\bar{A}_1 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}.$$

Therefore, only the column of x_1 in the simplex tableau will change. The coefficient of x_1 in Row (0) will become

$$\mathbf{c_B}^T \mathbf{B}^{-1} \bar{A}_1 - c_1 = [4 \ \ 1] \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} - 3 = -0.5 < 0,$$

and the column for x_1, except Row (0), is

$$\mathbf{B}^{-1} \bar{A}_1 = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}.$$

We have the following tableau which serves as a starting tableau for the simplex tableau.

Basic Variable	Row	Z	x_1	x_2	x_3	s_1	s_2	RHS	Ratios
Z	(0)	1	-0.5	0	0	4	1	300	
x_3	(1)	0	0.5	0	1	1.5	-0.5	25	$25/0.5 = 50 \leftarrow$min
x_2	(2)	0	0	1	0	-0.5	0.5	25	

Basic Variable	Row	Z	x_1	x_2	x_3	s_1	s_2	RHS	Ratios
Z	(0)	1	0	0	1	5.5	0.5	325	
x_1	(1)	0	1	0	2	3	-1	50	
x_2	(2)	0	0	1	0	-0.5	0.5	25	

Yes, the company should produce candy bar A.

7. Suppose the amount of candy bar D the company wants to produce is x_4. The LP becomes

$$\text{Maximize} \qquad Z = 3x_1 + 7x_2 + 5x_3 + 17x_4$$

such that

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & x_3 & + & 3x_4 & \leq & 50 \\ 2x_1 & + & 3x_2 & + & x_3 & + & 4x_4 & \leq & 100 \end{array}$$

and $x_1, x_2, x_3, x_4 \geq 0$. We have

$$c_4 = 17, \quad A_4 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

The coefficient for this new variable x_4 is

$$\mathbf{c_B}^T \mathbf{B}^{-1} A_4 - c_4 = [4 \ \ 1] \begin{bmatrix} 3 \\ 4 \end{bmatrix} - 17 = -1$$

(therefore the tableau is no longer optimal), and the column of x_4, except Row (0), is

$$\mathbf{B}^{-1}A_4 = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0.5 \end{bmatrix}.$$

We have obtained a new tableau

Basic Variable	Row	Z	x_1	x_2	x_3	x_4	s_1	s_2	RHS	Ratios
Z	(0)	1	3	0	0	-1	4	1	300	
x_3	(1)	0	0.5	0	1	2.5$*$	1.5	-0.5	25	$25/2.5 = 10 \leftarrow$min
x_2	(2)	0	0.5	1	0	0.5	-0.5	0.5	25	$25/0.5 = 50$

Basic Variable	Row	Z	x_1	x_2	x_3	x_4	s_1	s_2	RHS	Ratios
Z	(0)	1	3.2	0	0.4	0	4.6	0.8	310	
x_4	(1)	0	0.2	0	0.4	1	0.6	-0.2	10	
x_2	(2)	0	0.4	1	-0.2	0	-0.8	0.6	20	

Yes, the company should produce type D candy bar. $\square$

6 The duality theory in sensitivity analysis

The duality theory tells that: *if a set of basic variables $\mathbf{x_B}$ is feasible, it is optimal if and only if $(\mathbf{B}^{-1})^T\mathbf{c_B}$ is dual feasible.*

We will reconsider two cases in this section to illustrate how the dual theorem can be applied to study sensitivity analysis.

Case 1: Changing $c_j \to \bar{c}_j$ or $A_j \to \bar{A}_j$ for some non-basic variable x_j.

Case 2: Adding a new activity.

In each case, the change actually maintains the feasibility of the current set of basic variables. Furthermore, the $(\mathbf{B}^{-1})^T\mathbf{c_B}$ remains the same.

Therefore, the current set of basic variables is still optimal if $(\mathbf{B}^{-1})^T\mathbf{c_B}$ remains dual feasible.

Example (revisited): The dual LP of the previous LP is

$$\text{Minimize} \qquad W = 50y_1 + 100y_2$$

such that

$$\begin{array}{rcrcl} y_1 & + & 2y_2 & \geq & 3 \\ y_1 & + & 3y_2 & \geq & 7 \\ y_1 & + & y_2 & \geq & 5 \end{array}$$

and $y_1, y_2 \geq 0$. We read off from the optimal simplex tableau that the dual optimal solution is $(y_1^*, y_2^*) = \mathbf{c_B}^T\mathbf{B}^{-1} = (4, 1)$.

1. Re-do the first part of question 1.
2. Re-do question 6, but only check if $\mathbf{x_B}$ remains optimal.
3. Re-do question 7, but only check if $\mathbf{x_B}$ remains optimal.

Solution: All these changes will not change the feasibility of the set of basic variables $\mathbf{x_B}$, and it will not change the $(\mathbf{B}^{-1})^T \mathbf{c_B} = [4 \;\; 1]^T$.

1. Suppose now the profit for candy bar A is $\bar{c}_1$. Only the first dual constraint is going to change. Therefore, $\mathbf{x_B}$ will remain optimal if and only if $(y_1, y_2) = (4, 1)$ satisfies constraint

$$y_1 + 2y_2 \geq \bar{c}_1$$

   or equivalently $\bar{c}_1 \leq 6$.

2. Suppose A_1 now becomes $\bar{A}_1 = [0.5 \;\; 0.5]^T$. Only the first constraint is going to change. Therefore, $\mathbf{x_B}$ remains optimal if and only if $(y_1, y_2) = (4, 1)$ satisfies constraint

$$0.5y_1 + 0.5y_2 \geq 3, \quad \text{or} \quad 0.5 \cdot 4 + 0.5 \cdot 1 \geq 3.$$

   But the inequality does not hold, hence $\mathbf{x_B}$ is no longer optimal.

3. The introduction of the new variable will produce a new dual constraint

$$3y_1 + 4y_2 \geq 17.$$

   But for $(y_1, y_2) = (4, 1)$, this constraint is not satisfied. Therefore, $\mathbf{x_B}$ is no longer optimal. $\square$