Integer Programming

A **pure IP** (resp. **mixed IP**) is an LP in which all (resp. some) decision variables are required to be integers. An IP is said to be **binary** (BIP) if all decision variables can only take value 0 or 1.

If we omit all integer or 0-1 constraints on decision variables of an IP, we obtain a usual LP, which is called the **LP relaxation** of the IP. It is easy to see that the *feasible region of any IP must be contained in the feasible region for its LP relaxation.* This easily implies that

$$\text{optimal value for LP relaxation} \quad \geq \quad \text{optimal value for IP},$$

if the IP is a maximization problem. For a minimization IP, the inequality is reversed.

The LP relaxation can be solved efficiently via the simplex algorithm, which basically move from one BFS to another neighboring *better* BFS. Analogously, one would hope that an IP could be solved by an algorithm that proceeded from one feasible integer solution to a better feasible integer solution. Unfortunately, no such algorithm is known.

A not so unnatural approach is to solve the LP relaxation of an IP, then round-off the optimal solution, hoping the resulting integer solution is optimal for the IP. However, such an operation rarely yields an optimal solution.

Example: Consider the following IP problem:

$$\text{Minimize} \qquad Z = 3x_1 + 4x_2$$

under constraints

$$\begin{array}{rcrcl} 2x_1 & + & 2x_2 & \leq & 5 \\ 2x_1 & + & 3x_2 & \leq & 6 \end{array}$$

and $x_1, x_2 \geq 0$; x_1, x_2 integer.

The optimal solution for its LP relaxation is

$$(x_1^*, x_2^*) = (1.5, 1).$$

Its roundoff is either $(2, 1)$ which is infeasible, or $(1, 1)$ which is not optimal for the IP (indeed, the optimal solution for the IP is $x_1^* = 0, x_2^* = 2$).

1 Formulating IP

We have already seen examples of IP in the knapsack problem and the general resource allocation problem. Below is a simple example.

Example: A company is considering four investments. Investment 1, 2, 3, and 4 will yield \$16000, \$22000, \$12000, and \$8000, respectively. Each investment requires a certain capital at present time: Investment 1, \$5000; investment 2, \$7000; investment 3, \$4000; and investment 4, \$3000. At present, \$14000 in total is available. Formulate an IP to maximize the total yield.

Solution: For each investment, the choice is either invest or not. This leads us to define

$$x_j = \begin{cases} 1 & ; \text{ if investment } j \text{ is made} \\ 0 & ; \text{ otherwise} \end{cases}, \qquad j = 1, 2, 3, 4.$$

The total yield will then be (in thousand dollars)

$$Z = 16x_1 + 22x_2 + 12x_3 + 8x_4,$$

and the total cost (in thousand dollars) is

$$5x_1 + 7x_2 + 4x_3 + 3x_4.$$

Therefore, the IP is

$$\text{Maximize} \qquad Z = 16x_1 + 22x_2 + 12x_3 + 8x_4$$

such that

$$5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14$$

and $x_j = 0$ or 1 $(j = 1, 2, 3, 4)$

Example: Modify the IP from the previous example, for each of the following additional constraints.

1. The company can invest in at most 2 investments.
2. If the company invests in investment 2, they must also invest in investment 1.
3. If the company invests in investment 2, they cannot invest in investment 4.
4. The company must invest in one and only one of investment 2 and investment 4.
5. The company can invest in investment 2 only if the company has made an investment in investment 3.

Solution:

1. Simply add constraints
$$x_1 + x_2 + x_3 + x_4 \leq 2.$$
2. $x_1 \geq x_2$. In other words, when $x_2 = 1$, x_1 will have to be 1, and when $x_2 = 0$, x_1 can take either 0 or 1.
3. $x_2 + x_4 \leq 1$. In other words, if $x_2 = 1$, then $x_4 \leq 0$, which imposes that $x_4 = 0$, and if $x_2 = 0$, x_4 can take either o or 1.
4. $x_2 + x_4 = 1$. In other words, if $x_2 = 1$, then $x_4 = 0$, and if $x_2 = 0$, then $x_4 = 1$.
5. $x_2 \leq x_3$. In other words, if $x_3 = 0$, then $x_2 = 0$, and if $x_3 = 1$, x_2 can take either o or 1.

1.1 Fixed-charge problems

To undertake an activity, a fixed-charge or setup cost is often incurred. In this case, the decision is made in two phases: (1) Should we undertake the activity? If we decide “no”, the cost is zero; if we decide “yes”, a fixed-charge is incurred. (2) If the answer is “yes” to the first question, then we must decide to which level the activity is going to be implemented. This will incur an additional cost, which is assume to be linear.

Example: A company is capable of making three types of clothing: shirts, shorts, and pants. The manufacture of each type of clothing requires that the company have the appropriate type of machinery available. The machinery needed to manufacture each type of clothing must be rented at the following rates: shirt machinery, \$200 per week; shorts machinery, \$150 per week; pants machinery, \$100 per week. The manufacture of each type of clothing also requires the amounts of cloth and labor shown below. Each week, 150 hours of labor and 160 square yard cloth are available. Formulate an IP to maximize the weekly profit.

	Labor (hours)	Cloth (square yards)	Profit (dollars)
Shirt	3	4	6
Shorts	2	3	4
Pants	6	4	7

Solution: The company must decide how many of each type of clothings should be produced. Let

$$\begin{aligned} x_1 &= \text{number of shirts produced each week} \\ x_2 &= \text{number of shorts produced each week} \\ x_3 &= \text{number of pants produced each week.} \end{aligned}$$

Define

$$y_1 = \begin{cases} 1 & ; \text{ if any shirts are manufactured} \\ 0 & ; \text{ otherwise} \end{cases}$$

and similarly y_2 for shorts, y_3 for pants. In other words, if $x_i > 0$ then $y_i = 1$, and if $x_i = 0$ then $y_i = 0$.

We write down the following IP:

Maximize $\quad Z = (6x_1 + 4x_2 + 7x_3) - (200y_1 + 150y_2 + 100y_3)$

such that

$$\begin{aligned} 3x_1 + 2x_2 + 6x_3 &\leq 150 \\ 4x_1 + 3x_2 + 4x_3 &\leq 160, \end{aligned}$$

and $x_1, x_2, x_3 \geq 0$; x_1, x_2, x_3 integer; $y_1, y_2, y_3 = 0$ or 1.

The optimal solution is found to be $(x_1^*, x_2^*, x_3^*) = (30, 0, 10)$, and $(y_1^*, y_2^*, y_3^*) = (0, 0, 0)$.

This is a *wrong* answer! The optimal solution requires that the shirts and pants be produced without renting the corresponding machinery. We must modify the IP such that whenever $x_j > 0$, $y_j = 1$ must hold.

The following trick resolve the problem. Let M be a *very large* number, and we add the following constraints to the IP:

$$\begin{aligned} x_1 &\le My_1 \\ x_2 &\le My_2 \\ x_3 &\le My_3. \end{aligned}$$

To illustrate, if $x_j > 0$ then y_j has to equal 1. If $x_j = 0$ then the constraints yields $y_j \ge 0$, and since $y_j = 1$ is more costly, the optimal solution will choose $y_j = 0$. On the other hand, when $y_j = 0$, then we must have $x_j = 0$. The only difficulty remaining is that when $y_j = 1$ we have the constraint $x_j \le M$, which seems to restrict the possible values of x_j. However, if we take M to be at least as large as the maximum feasible value of any decision variable x_1, x_2, x_3, then this restriction is essentially a non-restriction. For this specific example, x_1 cannot exceed 40, x_2 cannot exceed 160/3, and x_3 cannot exceed 25. It suffices to take $M \ge 160/3$, say $M = 54$.

With these new constraints, we can solve for the (correct) optimal solution, which is

$$(x_1^*, x_2^*, x_3^*) = (0, 0, 25), \quad (y_1^*, y_2^*, y_3^*) = (0, 0, 1), \quad Z^* = 75.$$

Or the company should produce 25 pants each week.

1.2 Either-or constraints

An either-or constraint is a choice that must be made between two constraints:

$$\begin{aligned} \text{either} \quad f(x_1, x_2, \cdots, x_n) &\le b_1 \\ \text{or} \quad g(x_1, x_2, \cdots, x_n) &\le b_2. \end{aligned}$$

The solution to this case, again, relies on an *very large* constant M. Consider the following set of constraints:

$$\begin{aligned} f(x_1, x_2, \cdots, x_n) &\le b_1 + My \\ g(x_1, x_2, \cdots, x_n) &\le b_2 + M(1 - y) \\ y &= 0 \text{ or } 1. \end{aligned}$$

Here y is an *auxiliary decision variable* taking binary values.

1. $y = 1$: In this case, the second constraint is satisfied, whereas the first constraint becomes

   $$f(x_1, x_2, \cdots, x_n) \le b_1 + M,$$

   which is essentially a non-constraint, if M is big enough (say bigger than the maximum value of $f - b_1$ over the feasible region).

2. $y = 0$: similar. In this case, the first constraint is satisfied.

Example: Ford is considering manufacturing three types of autos: compact, midsize, and large. The resources required for, and the profit yielded by, each type of car are shown below. At present, 6000 tons of steel and 60000 hours of labor are available. For production of a type of car to be economically feasible, at least 1000 cars of that type must be produced. Formulate an IP to maximize Ford's profit.

	Steel (tons)	Labor (hours)	Profit (dollars)
Compact	1.5	30	2000
Midsize	3	25	3000
Large	5	40	4000

Solution: Let

$$\begin{aligned} x_1 &= \text{number of compact cars produced} \\ x_2 &= \text{number of midsize cars produced} \\ x_3 &= \text{number of large cars produced.} \end{aligned}$$

Then the goal is to maximize the profit (in thousand dollars)

$$\text{Maximize} \qquad Z = 2x_1 + 3x_2 + 4x_3.$$

such that

$$\begin{aligned} 1.5x_1 + 3x_2 + 5x_3 &\leq 6000 \\ 30x_1 + 25x_2 + 40x_3 &\leq 60000 \end{aligned}$$

and $x_1, x_2, x_3 \geq 0$; x_1, x_2, x_3 integer.

However, we also have to consider the constraints

1. Either $x_1 = 0$ or $x_1 \geq 1000$. Equivalently, either $x_1 \leq 0$ or $1000 - x_1 \leq 0$.
2. Either $x_2 = 0$ or $x_2 \geq 1000$. Equivalently, either $x_2 \leq 0$ or $1000 - x_2 \leq 0$.
3. Either $x_3 = 0$ or $x_3 \geq 1000$. Equivalently, either $x_3 \leq 0$ or $1000 - x_3 \leq 0$

These constraint can be formulated as

$$\begin{aligned} x_1 &\leq M_1 y_1 \\ 1000 - x_1 &\leq M_1(1 - y_1) \\ y_1 &= 0 \text{ or } 1 \end{aligned}$$

and

$$\begin{aligned} x_2 &\leq M_2 y_2 \\ 1000 - x_2 &\leq M_2(1 - y_2) \\ y_2 &= 0 \text{ or } 1 \end{aligned}$$

and

$$\begin{aligned} x_3 &\leq M_3 y_3 \\ 1000 - x_3 &\leq M_3(1-y_3) \\ y_3 &= 0 \text{ or } 1 \end{aligned}$$

here M_1, M_2, M_3 are very large numbers. Indeed, we can choose $M_1 = 2000, M_2 = 2000, M_3 = 1200$ (why?). □

A more general either-or constraints is as follows. Suppose we have N possible constraints

$$\begin{aligned} f_1(x_1, \cdots, x_n) &\leq b_1 \\ f_2(x_1, \cdots, x_n) &\leq b_2 \\ &\vdots \\ f_N(x_1, \cdots, x_n) &\leq b_N, \end{aligned}$$

and the requirement is that some K of these constraints must hold. The solution is to write

$$\begin{aligned} f_1(x_1, \cdots, x_n) &\leq b_1 + My_1 \\ f_2(x_1, \cdots, x_n) &\leq b_2 + My_2 \\ &\vdots \\ f_N(x_1, \cdots, x_n) &\leq b_N + My_N, \\ y_1 + y_2 + \cdots + y_N &= N - K, \\ y_j &= 0 \text{ or } 1, \ j = 1, \cdots, N. \end{aligned}$$

Here M is a very large number (at least as large as the maximum value of $f_n - b_n$ over the feasible region).

2 Branch and Bound method

The branch-and-bound technique is very useful for solving pure IP or mixed IP problems. Below, we will illustrate the main idea of the technique to solve BIP problems. It is based on a very elementary observation: *If the relaxed LP has an optimal solution in which all variables are 0 or 1, then the solution is also optimal to the BIP.*

Example: Let us consider the BIP from the investment problem.

$$\text{Maximize} \qquad Z = 16x_1 + 22x_2 + 12x_3 + 8x_4$$

such that .

$$5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14$$

and $x_j = 0$ or 1 $(j = 1, 2, 3, 4)$

1. Solve the relaxed LP to obtain an optimal solution

$$(x_1, x_2, x_3, x_4) = (1, 1, 0.5, 0), \qquad Z = 44.$$

   It is not optimal for the BIP since it is infeasible.

2. Divide the original problem into two subproblems according to $x_3 = 0$ (Subproblem 2) or $x_3 = 1$ (Subproblem 3). For example, Subproblem 3 is

$$\text{Maximize} \qquad Z = 16x_1 + 22x_2 + 12 + 8x_4$$

   such that

$$5x_1 + 7x_2 + 3x_4 \leq 10$$

   and $x_j = 0$ or 1 ($j = 1, 2, 4$)

   We arbitrarily choose to solve Subproblem 3 before Subproblem 2. The relaxed LP yields an optimal solution

$$(x_1, x_2, x_3, x_4) = (1, \frac{5}{7}, 1, 0), \qquad Z = 43\frac{5}{7}.$$

3. Proceed as before, the relaxed LP for Subproblem 4 yields an optimal solution

$$(x_1, x_2, x_3, x_4) = (1, 0, 1, 1), \qquad Z = 36.$$

   Since this optimal solution takes value in (binary) integers for all decision variables, it is optimal for the IP defined in Subproblem 4. This is also a **candidate** for the optimal solution for the original BIP. We also know that the true optimal value for the original problem is at least 36, or $Z^* \geq 36$. We will denote by $LB = 36$ (lower Bound)

4. Subproblem 6 yields another candidate optimal solution, with

$$(x_1, x_2, x_3, x_4) = (1, 0, 1, 1), \qquad Z = 42.$$

   This is better than the other candidate optimal solution. Therefore, the other candidate we have obtained is *not* optimal. And $LB = 42$.

5. Subproblem 7 is infeasible, since it requires $x_1 = x_2 = x_3 = 1$.

6. Now we go back to Subproblem 2, which can be divided into Subproblem 8 and Subproblem 9. Subproblem 8 yields an optimal value $38 < LB$, there it can never be optimal. Subproblem 9 yields an optimal solution

$$(x_1, x_2, x_3, x_4) = (1, \frac{6}{7}, 0, 1), \qquad Z = 42\frac{6}{7}.$$

   Note that the optimal value must be an integer for the original IP. Therefore, branching from subproblem 9 will never yield an optimal value larger than 42. Thus we eliminate Subproblem 9, and conclude that an optimal solution is

$$(x_1^*, x_2^*, x_3^*, x_4^*) = (1, 0, 1, 1), \qquad Z^* = 42$$