Introduction to Operations Research

The substantial development of operations research started in World War II, when the demand for the better, if not the best, allocation of the very limited resource was prevalent. The scientists involved were asked to do *research on operations*, hence came the name "operation research (OR)". After World War II, operations research found many applications outside the military as well. However, the complexity of the problem became so high that it was impossible to calculate the solution by hand. Then came the computer – and with it the *simplex method* of George Dantzig. The simplex method does for *linear programming* (a major part of OR) what Gauss's method of elimination does for linear algebra: it gives you a way to computer the answer. Dantzig's method appeared in 1951, and suddenly linear programming sprang to life.

1 Simple examples of linear programming, and graphical method

Example (investment management): In 1972 Alfred Broaddus wrote an article for the Monthly Review of the Federal Reserve Bank of Richmond. It is called "linear programming: A new approach to bank portfolio management". The purpose of this article is to explain linear programming to bankers — during the 1960's, the Bankers Trust Company had developed a complex linear programming model to help the managers reach their investment decisions. The model had proved useful, and so other bankers gots interested. A much simplied example was used by Broaddus to illustrate the main idea.

Suppose the bank has 100 million dollars. Part of this money will be put into loans, and part into securities. Loans earn high interest (say 10%). Securities earn lower interest (say 5%), but they are more liquid than the loans: at any time, the securities can be sold at market value.

Let L and S be the amount of money in loans and securities. Then the total rate of return is

$$Z \doteq 0.10L + 0.05S.$$

The bank want to maximize this rate of return subject to certain constraints:

1. *Sign constraint:* We must have
$$L \geq 0, \quad S \geq 0.$$
2. *Total-fund constraint:* Assuming that the total amount available is 100 (in millions of dollars), we must have
$$L + S \leq 100.$$

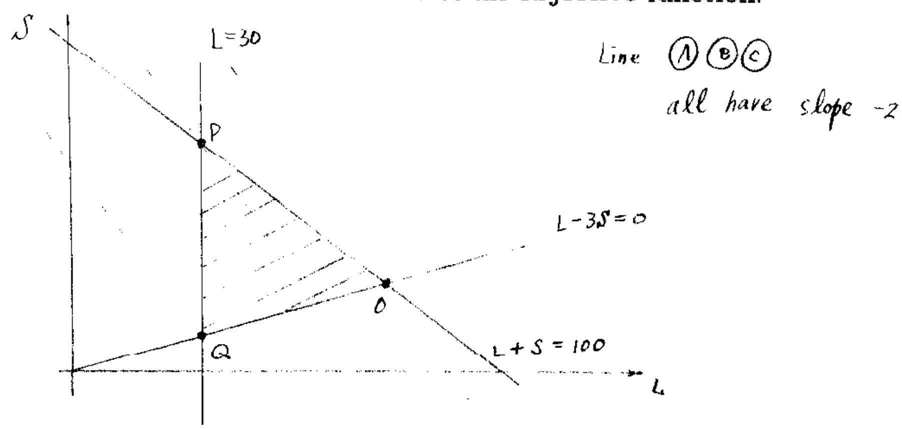
3. *Liquidity constraint:* For various reasons (Federal Reserve requirements, etc.), the bank wishes to keep at least 25% of its investment liquid. This means $S \geq 0.25(L + S)$, or equivalently,

$$L - 3S \leq 0.$$

4. *Loan-balance constraint:* The bank has certain big customers it never wants to disappoint. If they want loans, loans they shall have. The bank expects its primer customers to ask for loans totaling 30 million dollars, and so L must be at least that big:

$$L \geq 30.$$

We can solve this investment problem by the **graphical method**; i.e. pictures. Every point in the shaded region below stands for an allocation (L, S) that satisfies all the four constraints. We call every such point an **feasible solution**, and the shaded region the **feasible region**. The rate function Z we want to maximize is said to be the **objective function**.



Which point in the feasible region is optimal? To find out, we draw lines of constant return

$$0.10L + 0.05S = \text{constant}.$$

These are the lines with slope -2, hence they are parallel to each other. All points (L, S) on each of these lines give the same return. Draw the line of slope -2 through vertex Q, P, O respectively. This gives three parallel lines, with the least return on line through Q and the greatest return on line through O. All other points in the shaded region have intermediate returns. In other words, **the point O is an optimal solution**; indeed, in this case Q is the *unique* optimal solution. It is not difficult to see that $O = (75, 25) := (L^*, S^*)$, and the optimal rate of return is

$$Z^* = 0.10 \times L^* + 0.05 \times S^* = 0.10 \times 75 + 0.05 \times 25 = 8.75.$$

That is, the optimal annual rate of return is 8.75 million dollars.

As a check, note $Q = (30, 10)$ and $P = (30, 70)$, and they will produce an annual rate of return 3.5 and 6.5 million dollars, respectively. The are both suboptimal.

Exercise: Suppose that the total amount available for investment becomes 120 million, and the liquidity constraint changes to 50%. Use graphical method to solve for the new optimal allocation and the optimal annual rate of return.

Example (bond management): Linear programming models are used by many Wall street firms to select a desirable bond portfolio. Suppose that the firm has 10 million dollars available for the investment in two bonds; see the table below.

	Expected return	Worst-case return	duration
Bond 1	14%	6%	4 yrs
Bond 2	10%	10%	4 yrs

Define

$$\begin{aligned} x_1 &= \text{the amount invested in Bond 1} \\ x_2 &= \text{the amount invested in Bond 2} \end{aligned}$$

The firm wants to maximize the total expected return

$$Z = 0.14x_1 + 0.10x_2,$$

subject to the following constraints:

1. *Sign constraint:* We must have

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

2. *Total-fund constraint:* Assuming that the total amount available is 10 (in millions of dollars), we must have

$$x_1 + x_2 \leq 10.$$

3. *Worst-case constraint:* The worst case return of the Bond portfolio must be at least 8%. In other words, $0.06x_1 + 0.10x_2 \geq 0.08$, or equivalently,
   ↳ (x_1+x_2)

$$x_1 - x_2 \leq 0.$$

4. *Diversification constraint:* At most 80% of the total amount invested can be invested in a single bond. That is, $x_1 \leq 0.80(x_1 + x_2)$ and $x_2 \leq 0.80(x_1 + x_2)$; or

$$x_1 - 4x_2 \leq 0$$

and

$$x_2 - 4x_1 \leq 0.$$

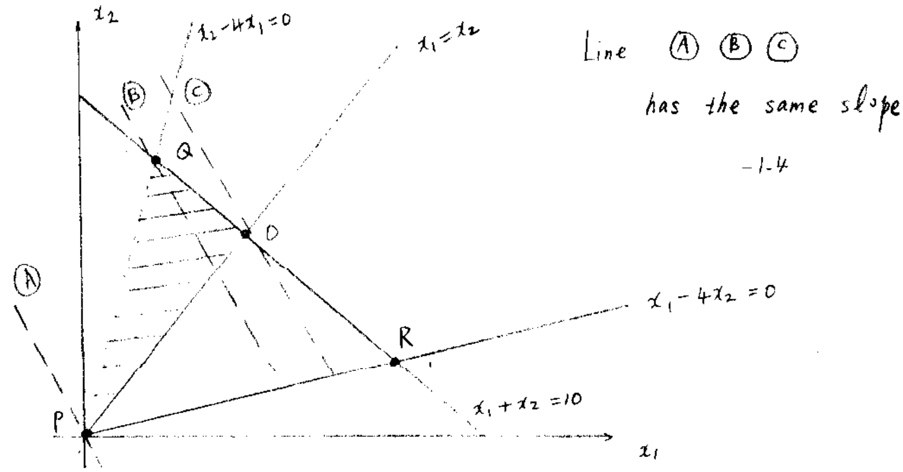
This LP (linear programming) problem can also be solved by the graphical method. As before, the shaded region consists of all the feasible solutions. Note in this case, the constraint $x_1 - 4x_2 \leq 0$ is redundant. Without this constraint, we would still obtain the same feasible region, whence the same optimal solution.

To find out the optimal solution, we draw the lines of constant return

$$0.14x_1 + 0.10x_2 = \text{constant}.$$

It is not very difficult to see that the optimal solution is point $O = (5,5) := (x_1^*, x_2^*)$, and the optimal expected return is

$$Z^* = 0.14 \times x_1^* + 0.10 \times x_2^* = 0.13 \times 5 + 0.10 \times 5 = 0.12.$$



Example (diet problem): What is the minimum cost for a nutrionally adequate diet? This famous diet problem serves as a perfect example of the standard minimum problem in LP. Below is a much simplied version.

Suppose a daily diet for a weighter-lifter consists of only milk and vegetable. The following table contain the cost and nutrition break-down for each food.

	Protein	Carbohydrate	Fat	Carolies	Cost
Skim milk	10gm	10gm	4gm	120	0.15
Vegetable salad	3gm	12gm	0gm	60	0.12

Assume that 1gm protein = 4 carolies; 1gm carbo = 4 carolies; and 1gm fat = 9 carolies. Suppose

$$x_1 \doteq \text{amount of milk in the daily diet}$$
$$x_2 \doteq \text{amount of salad in the daily diet}$$

The lifter wants to minimize the cost of the daily diet; i.e. the objective is to

$$\text{Minimize} \quad Z \doteq 0.15x_1 + 0.12x_2$$

subject to the following constraints.

1. *Sign constraint:* We must have

$$x_1 \geq 0; \quad x_2 \geq 0.$$

2. *Total-intake constraint:* The weighter-lifter, at body-weight 150 pounds, needs at least 2400 carolies a day. That is $120x_1 + 60x_2 \geq 2400$, or

$$2x_1 + x_2 \geq 40.$$

3. *Fat-intake constraint:* Fat intake should be at most 25%; i.e. $9{\cdot}4x_1 \leq 0.25(120x_1 + 60x_2)$. Or equivalently

$$2x_1 - 5x_2 \leq 0.$$

4. *Protein-intake constraint:* Protein intake should be at least 30%; i.e. $4 \cdot 10x_1 + 4 \cdot 3x_2 \geq 0.30(120x_1 + 60x_2)$. Or equivalently

$$2x_1 - 3x_2 \geq 0.$$

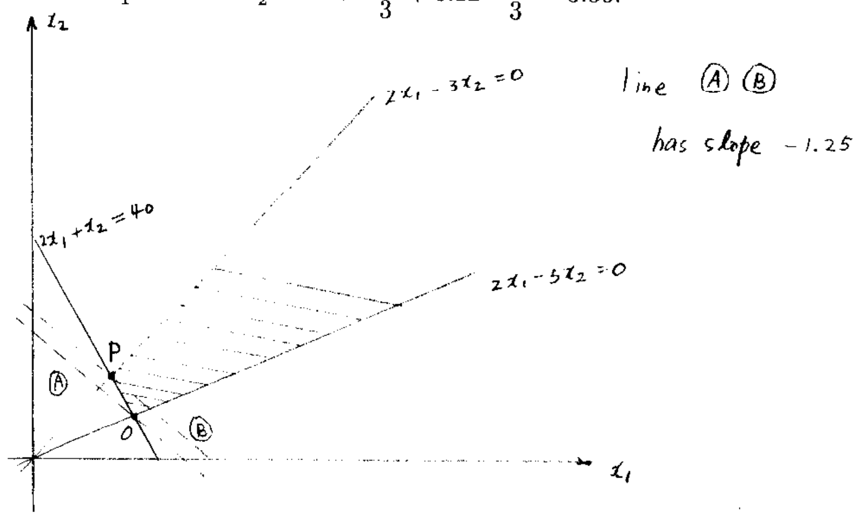
This LP (linear programming) problem can also be solved by the graphical method. As before, the shaded region consists of all the feasible solutions.

To find out the optimal solution, we draw the lines of constant return

$$0.15x_1 + 0.12x_2 = \text{constant}.$$

It is not very difficult to see that the optimal solution is point $O = (50/3, 20/3) := (x_1^*, x_2^*)$, and the minimal daily cost is

$$Z^* = 0.15 \times x_1^* + 0.12 \times x_2^* = 0.15 \cdot \frac{50}{3} + 0.12 \cdot \frac{20}{3} = 3.30.$$



2 More complicated examples

Example (diet problem): In 1945, George Stigler published a paper called "The cost of subsistence." It presented a basic economic problem of world food supply and appeared in Journal of Farm economics.

Suppose we label the available foods $1, 2, \cdots, n$. A daily diet for a single individual is a set of components $x_1, x_2, \cdots, x_n$; for instance, x_3 would be the amount of third food in the daily diet of this individual.

If the unit price of food j costs c_j, then the component x_j costs $c_j x_j$. We want to minimize the total cost

$$Z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n := \sum_{j=1}^{n} c_j x_j.$$

An adequate diet must provide at least the minimum daily requirements of certain nutrients – calories, vitamins, protein, fat, carbohydrate, fiber, minerals, etc. The available foods are known to contain the required nutrients in various amounts. Likewise, we label the nutrients by $i - 1, 2, \cdots, m$.

Let a_{ij} be the amount of nutrient i in one unit of foof j. The total amount of nutrient i provided by the diet is

$$a_{i1}x_1 + a_{i2}x_2 + \cdots a_{in}x_n.$$

Let b_i be the minimum daily requirement for nutrient i. Then an adequate diet must satisfy the constraint

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i, \qquad i = 1, 2, \cdots, m;$$

and sign constraint

$$x_1 \geq 0, \quad x_2 \geq 0, \quad \cdots, \quad x_n \geq 0.$$

Under those constraints, we wish to minimize the cost

$$\text{Minimize} \quad Z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n.$$

In Stilger's paper, he propsed a diet problem with 77 foods (i.e., $n = 77$) and 10 nutritional requirements (i.e., $m = 10$) that have to be satisfied. When solved by computer, the optimal solution yields a diet consisting of corn meal, wheat flour, evaporated milk, peanut butter, lard, beef, liver, potatoes, spinach, and cabbage. Even though such a diet is clearly highly nutritional, but it does not seem to meet a minimum standard tastiness.

Example (work scheduling in post office): A post office requires a different number of hours of labor in different days of week. The number of hours required on each day is given below.

	Hours of labor required
Monday	136
Tuesday	104
Wednesday	120
Thursday	152
Friday	112
Saturday	128
Sunday	88
Total	840

The post office may meet its daily labor requirement by hiring full-time or part-time employees. Each full-time employee (FTE) works 8 hours a day for 5 consecutive days, and take 2 days off; and a part-time employee (PTE) does the same except 4 hours a day for 5 consecutive days.

A full time employee costs the post office \$15 per hour, whereas a part-time employee costs only \$10 per hour. Union requirements limit part-time labor to 25% weekly labor cost.

How should we formulate an LP to minimze the weekly labor cost?

The correct formulation of this problem use the following variables:

$$
\begin{aligned}
x_1 &\doteq \text{Number of FTE who } \textbf{start} \text{ work on Monday} \\
&\vdots \\
x_7 &\doteq \text{Number of FTE who } \textbf{start} \text{ work on Sunday} \\
y_1 &\doteq \text{Number of PTE who } \textbf{start} \text{ work on Monday} \\
&\vdots \\
y_7 &\doteq \text{Number of PTE who } \textbf{start} \text{ work on Monday}
\end{aligned}
$$

The objective is to

$$
\text{Minimize} \qquad Z \doteq 15 \cdot 8 \cdot 5 \sum_{j=1}^{7} x_j + 10 \cdot 4 \cdot 5 \sum_{j=1}^{7} y_j
$$

subject the constraints

$$
\begin{aligned}
8(x_1 + x_4 + x_5 + x_6 + x_7) + 4(y_1 + y_4 + y_5 + y_6 + y_7) &\geq 136 \\
8(x_1 + x_2 + x_5 + x_6 + x_7) + 4(y_1 + y_2 + y_5 + y_6 + y_7) &\geq 104 \\
8(x_1 + x_2 + x_3 + x_6 + x_7) + 4(y_1 + y_2 + y_3 + y_6 + y_7) &\geq 120 \\
8(x_1 + x_2 + x_3 + x_4 + x_7) + 4(y_1 + y_2 + y_3 + y_4 + y_7) &\geq 152 \\
8(x_1 + x_2 + x_3 + x_4 + x_5) + 4(y_1 + y_2 + y_3 + y_4 + y_5) &\geq 112 \\
8(x_2 + x_3 + x_4 + x_5 + x_6) + 4(y_2 + y_3 + y_4 + y_5 + y_6) &\geq 128 \\
8(x_3 + x_4 + x_5 + x_6 + x_7) + 4(y_3 + y_4 + y_5 + y_6 + y_7) &\geq 88 \\
20(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7) &\leq 0.25 \times 840
\end{aligned}
$$

and

$$
x_i \geq 0, \quad y_i \geq 0; \qquad i = \{1, 2, \cdots, 7\}.
$$

Remark: The above example can be easily formulated in the wrong way. It is very appealing to formulate the LP as follows:

$$
\begin{aligned}
x_1 &\doteq \text{Number of FTE who work on Monday}\\
&\vdots\\
x_7 &\doteq \text{Number of FTE who work on Sunday}\\
y_1 &\doteq \text{Number of PTE who work on Monday}\\
&\vdots\\
y_7 &\doteq \text{Number of PTE who work on Monday}
\end{aligned}
$$

and the objective is

$$\text{Minimize} \qquad Z \doteq 15 \cdot 8 \sum_{j=1}^{7} x_j + 10 \cdot 4 \sum_{j-1}^{7} y_j$$

subject to constraints

$$
\begin{aligned}
8x_1 + 4y_1 &\geq 136\\
8x_2 + 4y_2 &\geq 104\\
8x_3 + 4y_3 &\geq 120\\
8x_4 + 4y_4 &\geq 152\\
8x_5 + 4y_5 &\geq 112\\
8x_6 + 4y_6 &\geq 128\\
8x_7 + 4y_7 &\geq 88\\
4(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7) &\leq 0.25 \times 840
\end{aligned}
$$

and

$$x_i \geq 0, \quad y_i \geq 0; \qquad i = \{1, 2, \cdots, 7\}.$$

This is wrong because the variables $x_1, x_2, \cdots, x_7$ defined here are interrelated and the interrelation is not captured by the constraints. For example, the choice

$$(x_1, \cdots, x_7) = (2000, 20, 20, 20, 20, 20, 20), \qquad (y_1, \cdots, y_7) = (0, \cdots, 0)$$

is in the feasible region, but it is impossible to achieve this choice.

Remark: A problem here is that the optimal solution to the LP problem might not take values in integers. However, we can always round the numbers if necessary, or resort to *integrer programming*, which will always yield all-integer solution.

Example (Galaxy population): In 1972, Harvard astronomer S.M. Faber published a paper in Astronomy and Astrophysics called "Quadratic programming applied to the problem of galaxy population synthesis." She wanted to find out the number of stars in various galaxies.

She sucessfully reduced the population problem into a *least-square* calculation, and her problem looked something like this

$$\text{Minimize} \qquad \sum_{i=1}^{m} (a_{i1}x_1 + \cdots + a_{in}x_n - b_i)^2$$

When she made the computation, some of the galaxy populations came out negative. Bad. Galaxy populations are never negative. Hence sign constraints are put into consideration, and she required

$$x_1 \geq 0, \quad \cdots, \quad x_n \geq 0.$$

With the sign constraints, the problem is an example of the so-called **quadratic programming**.

Quadratic programming is a **non-linear programming**. You have no right to hope that it can be solved by the simplex method of linear programming. However, it can, as shown by Philip Wolfe in an article appear in the economic journal Econometrica.

Example (Chebyshev approximation): This is a marvelous example that a seemingly non-linear problem can be formulated as an LP.

We are given an over-determined system of linear equations

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \qquad (i = 1, \cdots, m);$$

here $m > n$, or there are more equations than unknowns. We cannot expect to solve these equations exactly.

Inevitably, there must be errors. Let

$$\varepsilon_i \doteq b_i - (a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n) \qquad (i = 1, \cdots, m).$$

The errors (ε_i) will depend on our choice of (x_j). Let us define the maximum absolute error

$$\mu \doteq \max\{|\varepsilon_1|, \ |\varepsilon_2|, \ \cdots, \ |\varepsilon_n|\}$$

The objective is to choose (x_j) so as to

$$\text{Minimize} \qquad \mu.$$

The qualitative property (existence and uniqueness) of Chebyshev approximation was well known. But no one knew how to compute the answers, until Edward Stiefel proposed an equivalent LP around 1960. His idea is as follows:

The maximum absolute error μ satisfies

$$-\mu \leq \varepsilon_i \leq \mu, \qquad \forall\ i = 1, 2, \cdots, m.$$

In other words, μ satisfies inequalities

$$-\mu \leq b_i - (a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n) \leq \mu, \qquad \forall\ i = 1, 2, \cdots, m.$$

It is not hard to see that the original problem is equivalent to the following LP problem: Define a new unknown x_0 (which is essentially μ), the objective is to

$$\text{Minimize} \qquad x_0$$

under the constraints

$$\begin{aligned} x_0 + a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n &\geq b_i \\ x_0 - a_{i1}x_1 - a_{i2}x_2 - \cdots - a_{in}x_n &\geq -b_i, \qquad \forall\, i = 1, 2, \cdots, m. \end{aligned}$$

Exercise: State the following problem as a linear program:

$$\text{Minimize} \qquad 5 \cdot |3x_1 + 4x_2 - 7| + 2 \cdot |2x_1 + 3x_2 - 5| + 8 \cdot |-x_1 + 4x_2 - 9|;$$

here x_1, x_2 can take arbitrary real value.

3 Stanrdard form, canonical form and their equivalence

We will now formulate the general linear programming model. A general LP problem is an optimization (minimization or maximization) problem where the *objective function* (the quantity to be optimized) are linear functions and the constraints are linear inequalities (or equalities).

3.1 Standrad form

The following is called the **standard form** for LP.

Objective function: Select the value for $x_1, x_2, \cdots, x_n$ so as to

$$\text{Maximize} \qquad Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Functional constraints:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m \end{aligned}$$

Non-negativity constraints:

$$x_1 \geq 0, \quad x_2 \geq 0, \quad \cdots, \quad x_n \geq 0.$$

Of course, sometimes an LP problem will not immediately fit into the standard form (as we have already observed in some previous examples), but it will always, after some transformation. Following are several examples.

1. The objective function is to minimize:

$$\text{Minimize} \qquad Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n.$$

This is equivalent to the maximization problem

$$\text{Maximize} \qquad \bar{Z} = \bar{c}_1x_1 + \bar{c}_2x_2 + \cdots + \bar{c}_nx_n,$$

where

$$\bar{Z} = -Z, \quad \bar{c}_i = -c_i; \qquad \forall \ i = 1, 2, \cdots, n.$$

2. Some functional constraints might take form

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i,$$

for some i. It is equivalent to the constraints

$$\bar{a}_{i1}x_1 + \bar{a}_{i2}x_2 + \cdots + \bar{a}_{in}x_n \leq \bar{b}_i$$

where

$$\bar{b}_i = -b_i, \quad \bar{a}_{ij} = -a_{ij}; \qquad \forall \ j = 1, 2, \cdots, n.$$

3. Some functional constraints might take form

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i,$$

for some i. It is equivalent to the combination of two constraints

$$\begin{aligned} a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n &\leq b_i \\ \bar{a}_{i1}x_1 + \bar{a}_{i2}x_2 + \cdots + \bar{a}_{in}x_n &\leq \bar{b}_i \end{aligned}$$

where $\bar{b}_i$ and $\bar{a}_{ij}$ is defined as before.

4. Some variable, say x_i, might be unstricted in sign. In other words, x_i can take any real value. What we can do is to create two new variables u_i, v_i, and replace x_i everywhere by

$$x_i = u_i - v_i,$$

and add sign constraints $u_i \geq 0, \ v_i \geq 0$.

3.2 Canonical form

It is also convenient to define the following **canonical form** for LP problems.

Objective function: Select the value for $x_1, x_2, \cdots, x_n$ so as to

$$\text{Maximize} \qquad Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Functional constraints:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Non-negativity constraints:

$$x_1 \geq 0, \quad x_2 \geq 0, \quad \cdots, \quad x_n \geq 0.$$

The only difference between canonical form and standard form is that all the functional constraints are linear equalities instead of inequalities. From the discussion above, we know every LP in canonical form can be written as an LP in standard form. Indeed, the reverse is also true; i.e. every LP in standard form can be written as an LP in canonical form.

Theorem: Standard form and canonical form are equivalent.

Proof: It remains to show that any LP in standard form can be converted into an LP in canonical form. To this end, we will introduce some "slack" variables. The sole purpose of these variables is to make the functional constraints equalities. Indeed, introduce variables $x_{n+1}, \cdots, x_{n+m}$, and an LP in standard form can be written into the canonical form

Objective function: Select the value for $x_1, x_2, \cdots, x_n, x_{n+1}, \cdots, x_{n+m}$ so as to

$$\text{Maximize} \qquad Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n + c_{n+1}x_{n+1} + \cdots + c_{n+m}x_{n+m}$$

where $c_{n+1} = \cdots = c_{n+m} \doteq 0$.

Functional constraints:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \quad +x_{n+1} \qquad\qquad\qquad\qquad &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \qquad\qquad +x_{n+2} \qquad\qquad &= b_2 \\ \vdots \qquad\qquad\qquad\qquad \ddots \quad &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \qquad\qquad\qquad\qquad +x_{n+m} &= b_m \end{aligned}$$

Non-negativity constraints:

$$x_1 \geq 0, \quad x_2 \geq 0, \quad \cdots, \quad x_n \geq 0; \quad x_{n+1} \geq 0, \quad \cdots, \quad x_{n+m} \geq 0.$$

Note, the addition of slack variables will not affect the optiomal value and optimal solutions. □

Exercise: Convert the following general LP problem into the canonical form.

$$\text{Minimize} \qquad Z = 2x_1 + 3x_2 - x_3$$

subject to constraints

$$\begin{aligned} x_1 + x_2 &\leq 10 \\ 2x_1 - x_2 + 5x_3 &\leq 20 \\ -x_1 + x_2 + x_3 &= 10 \end{aligned}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \in \mathsf{R}.$$

Exercise: Furnco manufactures tables and chairs. A table requires 40 boar ft of wood, and a chair requires 30 board ft of wood. Wood may be purchased ar a cost of \$1 per board ft and 40000 board ft of wood are available in total. It takes 2 hours of skilled labor to manufacture an unfinished table or an unfinished chair. Three more hours of skilled labor will turn an unfinished table into a finished table, while 2 more hours of skilled labor will turn an unfinished chair into a finished chair. A total of 60000 skilled labor are available (and have already been paid for). All furniture produced can be sold at the following unit price: unfinished table \$70; finisished table \$140; unfinished chair \$60; finished chair \$110. Formulate an LP that will maximize the profit, and write it in the canonical form.

Solution: Define the following variable

$$\begin{aligned} x_1 &= \text{number of unfinished tables} \\ x_2 &= \text{number of finished tables} \\ x_3 &= \text{number of unfinished chairs} \\ x_4 &= \text{number of finished chairs} \\ x_5 &= \text{borad ft of wood purchased.} \end{aligned}$$

Furnaco want to maximize the profit, or the objective function is

$$\text{Maximize} \qquad \text{Profit} = \text{Sales} - \text{Cost} = 70x_1 + 140x_2 + 60x_3 + 110x_4 - x_5$$

under the constraints

$$\begin{aligned} x_5 &\leq 40000 \\ 2x_1 + 5x_2 + 2x_3 + 4x_4 &\leq 60000 \\ 40(x_1 + x_2) + 30(x_3 + x_4) - x_5 &\leq 0 \end{aligned}$$

and sign constraint

$$x_1 \geq 0, \quad \cdots, \quad x_5 \geq 0.$$

To write this as into canonical form, introduce slack variable x_6, x_7, x_8, and the LP becomes:

Objective funtion:

$$\text{Maximize} \qquad Z = 70x_1 + 140x_2 + 60x_3 + 110x_4 - x_5$$

Functional constraints:

$$\begin{array}{rcl} x_5 \quad +x_6 & & = 40000 \\ 2x_1 + 5x_2 + 2x_3 + 4x_4 \quad +x_7 & & = 60000 \\ 40(x_1 + x_2) + 30(x_3 + x_4) - x_5 \quad +x_8 & & = 0 \end{array}$$

and sign constraint

$$x_1 \geq 0, \quad \cdots, \quad x_5 \geq 0; \quad x_6 \geq 0, \quad x_7 \geq 0, \quad x_8 \geq 0.$$

Example: Use graphical method to solve the LP problem

$$\text{Minimize} \qquad Z = x_1 + x_2$$

subject to the constraints

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ 2x_1 + x_2 + x_4 &= 8 \end{aligned}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0.$$

Solution: Regard x_3, x_4 as slack variables, and the LP can be written as

$$\text{Minimize} \qquad Z = x_1 + x_2$$

subject to the constraints

$$\begin{aligned} x_1 + 2x_2 &\geq 4 \\ 2x_1 + x_2 &\leq 8 \end{aligned}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0.$$

The graphical method show that the optimal solution for this LP is

$$x^* = (x_1^*, x_2^*) = (0, 2) \qquad Z^* = x_1^* + x_2^* = 2.$$

It is not difficult to see that for the original LP problem, the optimal solution is

$$(x_1^*, x_2^*, x_3^*, x_4^*) = (0, 2, 0, 6) \qquad Z^* = 2.$$

3.3 Terminologies and pathological cases

- *decision variable:* the vector $(x_1, x_2, \cdots, x_n)$.
- *feasible solution:* any vector $(x_1, x_2, \cdots, x_n)$ that satisfies all the constraints.
- *feasible region:* the collection of all feasible solutions.

- *optimal solution:* a feasible solution that maximize (or minimize) the objective function.
- *most favorable value:* the largest (or smallest) possible value of the objective function.

It is possible that the feasible region is empty (i.e., no feasible solution). For example,

$$\text{Minimze} \qquad Z = x_1 + x_2$$

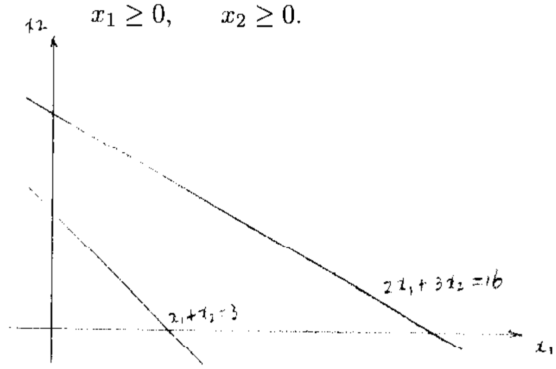
under constraints

$$\begin{aligned} x_1 + x_2 &\leq 3 \\ 2x_1 + 3x_2 &\geq 16 \end{aligned}$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

See the following graph.



It is also possible that an LP has non-empty feasible region, but no optimal solution. For example,

$$\text{Minimze} \qquad Z = -2x_1$$

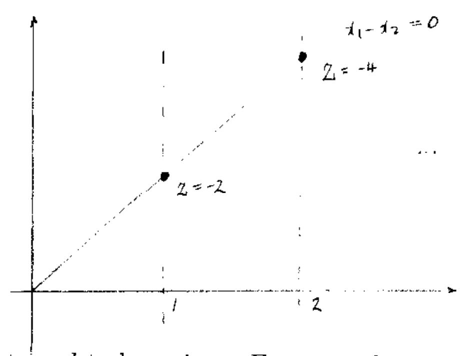
under constraints

$$x_1 - x_2 = 0$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

See the following graph.



If an optimal solution exists, it might not need to be unique. For example,

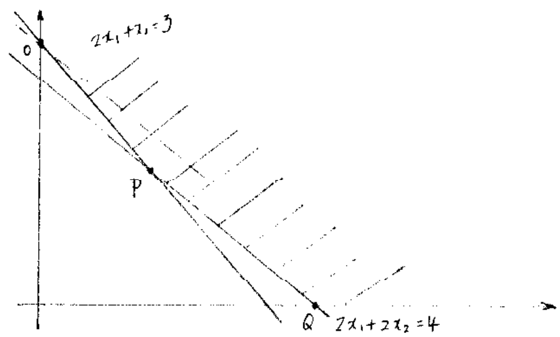
$$\text{Minimze} \qquad Z = x_1 + x_2$$

under constraints

$$\begin{aligned} 2x_1 + 2x_2 &\geq 4 \\ 2x_1 + x_2 &\geq 3 \end{aligned}$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$



Conclusion: We are not interested in these pathological cases. The *normal* case is this: An LP problem has many feasible solutions, among which one is the unique optimal solution.

4 Why not calculus

After all, LP problems are constrained maximization (or minimization) problems. From calculus, we know many such problems can be solved by *Lagrange multipliers method.* The question is why this methodology does not apply to LP problems. We should study the following example of LP.

Example: The objective is to

$$\text{Minimize} \qquad Z = 3x_1 + 5x_2$$

with constraints

$$2x_1 + x_2 = 4, \qquad x_i \geq 0; \quad i = 1, 2.$$

It is not difficult to check from the graphical method that the optimal solution is

$$(x_1^*, x_2^*) - (2, 0).$$

Now let us the Lagrange multiplier method: introduce the multiplier λ (unknown constant) and form the Lagrange function

$$\phi(x_1, x_2) = Z - \lambda(2x_1 + x_2 - 4) = 3x_1 + 5x_2 - \lambda(2x_1 + x_2 - 4).$$

Set the derivatives equal to zero:

$$\frac{\partial}{\partial x_1}\phi = 0, \qquad \frac{\partial}{\partial x_2}\phi = 0;$$

or equivalently,

$$3 - 2\lambda = 0, \qquad 5 - \lambda = 0.$$

Evidently, there is no multiplier that will satisfy both equations simultaneously.

Discussion: The reason that calculus does not work here is that: Calculus works when the optimal solution lies in the *interior* of the region. However, in all the examples of LP we have seen, the optimal solution is on the *boundary* on the feasible region.

This is indeed a general phenomenon. Think about maximizing (or minimizing) a linear function over a polyhedron, the optimal solution is always on the boundary. Therefore, calculus is not a suitable tool for studying LP.

5 Linear algebra: a brief review

The introduction of vectors, matrices and linear algebra sometimes is very useful in the study of LP. We only need a very small part of linear algebra, and it is summarized below.

A *vector* x is a collection of real numbers $x_1, x_2, \cdots, x_n$. The number x_i is the i-th component of vector x. One can regard x as a point (or a vector) in space R^n. We should use the standard notation, as write x as a *column vector*, or

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The corresponding *row vector* is nothing but the *transpose* of the column vector x, or

$$x^T = [x_1, \; x_2, \; \cdots, \; x_n].$$

A column vector is a special matrix with only one column; while a row vector is a special matrix with only one row.

A general $m \times n$ matrix takes the following form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Sometimes, we write $A = (a_{ij})$. The *transpose* of A is a $n \times m$ matrix defined by

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.$$

For example,

$$A = \begin{bmatrix} 0 & -\pi & \sqrt{2} \\ 1 & -2 & \sqrt{5} \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} 0 & 1 \\ -\pi & -2 \\ \sqrt{2} & \sqrt{5} \end{bmatrix}.$$

It is not difficult to see that

$$(A^T)^T = A$$

Operations of Matrices: Suppose $A = (a_{ij})$ is a $m \times n$ matirx.

1. *product with scalar:* If λ is a real number, then $\lambda A \doteq (\lambda a_{ij})$.
2. *summation:* suppose $B = (b_{ij})$ is another $m \times n$ matrix, then

   $$A + B \doteq (c_{ij}), \qquad c_{ij} \doteq a_{ij} + b_{ij}, \quad \forall\ i = 1, \cdots, m;\ j = 1, \cdots, n.$$

   It is not difficult to check that

   $$(A + B)^T = A^T + B^T.$$

   Note if B is not a $m \times n$ matrix, $A + B$ is not well defined.
3. *product with matrix:* Suppose $B = (b_{kl})$ is a $n \times r$ matrix, then AB is defined as a $m \times r$ matrix, with components

   $$c_{ij} \doteq \sum_{k=1}^{n} a_{ik} b_{kj}, \quad \forall\ i = 1, \cdots, m;\ j = 1, \cdots, r.$$

   The matrix product is *associative* but generally not commutative:

   $$(AB)C = A(BC), \quad \text{but} \quad AB \neq BA \ \ \text{(usually)}.$$

   Furthermore,

   $$(AB)^T = B^T A^T.$$
4. *inner product of two vectors:* If $x = (x_i)$ and $y = (y_i)$ are two $n \times 1$ column vectors, then their inner product is defined as

   $$\langle x, y \rangle \doteq x^T y = \sum_{i=1}^{n} x_i y_i = y^T x.$$
5. *inverse:* Suppose A is a $n \times n$ matrix. We say A is *invertible* if there exists a $n \times n$ matrix B such that,

   $$AB = BA = I_n;$$

   here I_n is the $n \times n$ identity matrix, or

   $$I_n \doteq \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

   We say B is the *inverse* of A, and denote by $B := A^{-1}$.

A collection of $n \times 1$ column vectors $x^{(1)}, x^{(2)}, \cdots, x^{(k)}$ is said to be *linearly independent* if no one of them can be written as a linear combination of the others. That means,

$$\sum_{j=1}^{k} \lambda_j x^{(j)} = 0 \quad \text{if and only if} \quad \lambda_j = 0 \ \text{ for all } j.$$

For example

$$x = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 \\ 6 \\ -9 \\ 0 \end{bmatrix}$$

are linearly independent. However, x, y, z are linear dependent, where

$$z = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

since z is a linear combination of x and y; indeed,

$$z = \frac{2}{3}(3x + y) = 2x + \frac{2}{3}y.$$

Let A be a $m \times n$ matrix. If A has r independent columns, but does not have $r+1$ independent columns, then we say

$$\text{rank of matrix } A = r := \text{rank}(A).$$

A theorem says that

$$\text{rank}(A) = \text{rank}(A^T)$$

for all matrix A.

The followinf results are stated without proof.

Theorem: Suppose A is a $n \times n$ matrix. Then the linear equations $Ax = b$ has a unique solution $x \in \mathsf{R}^n$ for every $b \in \mathsf{R}^n$ if and only if A has independent columns; i.e. $\text{rank}(A) = n$, and if and only if A is invertible.

Theorem: If A is a $m \times n$ matrix, and b is an $m \times 1$ vector, then the equation $Ax = b$ has some solution $x \in \mathsf{R}^n$ if and only if

$$\text{rank}(A) = \text{rank}[A,\ b].$$

For example, the equation

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} x = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

has a solution because

$$\text{rank} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix} = 1.$$

We will adopt the following notation: suppose x, y are two $n \times 1$ vectors, we say

$$x \geq y, \quad \text{if} \quad x_i \geq y_i, \quad \forall \ i = 1, \cdots, n.$$

Now we can rewrite the srandard form and the canonical form of LP in the more compact matrix form.

Standard form of LP: The objective is to choose a vector x so as to

$$\text{Maximize} \qquad Z = c^T x$$

subject to the constraints

$$Ax \leq b, \quad x \geq 0.$$

Here we use notation

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Canonical form of LP: The objective is to choose a vector x so as to

$$\text{Maximize} \qquad Z = c^T x$$

subject to the constraints

$$Ax = b, \quad x \geq 0.$$

Here x, A, b, c are similarly defined as in the standard form.

Below are some exercises about linear algebra.

Exercise: Suppose A is $m \times n$ matirx. Show that $\text{rank}(A) \leq \min\{m, n\}$. Hint: use definition and the fact that $\text{rank}(A) = \text{rank}(A^T)$.

Exercise: Suppose matrix

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

1. What is the rank of A? Verify that $\text{rank}(A) = \text{rank}(A^T)$
2. Compute AA^T and $A^T A$. Are they the same?
3. Show that for all $b \in \mathsf{R}^2$, the equation $Ax = b$ has some solution $x \in \mathsf{R}^3$.

Exercise: State the following LP into canonical form (introduce slack variable if necessary), and further express it in matrix form.

The objective is to

$$\text{Maximize} \qquad x_2$$

subject to the constraints

$$\begin{aligned} 3x_1 - 4x_2 &\geq 0 \\ 5x_1 + 2x_3 &\leq 0 \\ 6x_2 + 7x_3 &= 0 \end{aligned}$$

and sign constraints (only on x_1 and x_3)

$$x_1 \geq 0, x_3 \geq 0.$$