AM 121: Homework # 6 (solution)

1. One can regard this as a general resource allocation problem (note that different formulations can be applied to this problem). Let $1+x_n$ be the days assigned to course n, for $n = 1, 2, 3, 4$. Then we face the allocation problem

$$\text{Maximize} \quad \sum_{n=1}^{4} r_n(1+x_n)$$

such that

$$\sum_{n=1}^{4} x_n \leq 3, \qquad x_n \text{ is non-nagative integer.}$$

Here r_n is the grade gain for course n. Let

$$V_j(w) \doteq \max_{\{x_n\}} \sum_{n=j}^{4} r_n(1+x_n)$$

such that

$$\sum_{n=j}^{4} x_n \leq w.$$

We have

$$V_4(0) = r_4(1) = 6, \quad V_4(1) = r_4(2) = 7, \quad V_4(2) = r_4(3) = 9, \quad V_4(3) = r_4(4) = 9.$$

The DPE is

$$V_j(w) = \max_{\{0 \leq x \leq w\}} \left[r_j(1+x) + V_{j+1}(w-x) \right].$$

We can recursively compute

$$\begin{aligned}
V_3(0) &= 2+6 = 8 & (x^* = 0) \\
V_3(1) &= \max[2+7,\ 4+6] = 10 & (x^* = 1) \\
V_3(2) &= \max[2+9,\ 4+7,\ 7+6] = 13 & (x^* = 2) \\
V_3(3) &= \max[2+9,\ 4+9,\ 7+7,\ 8+6] = 14 & (x^* = 2, 3)
\end{aligned}$$

and

$$\begin{aligned}
V_2(0) &= 5+8 = 13 & (x^* = 0) \\
V_2(1) &= \max[5+10,\ 6+8] = 15 & (x^* = 0) \\
V_2(2) &= \max[5+13,\ 5+10,\ 6+8] = 18 & (x^* = 0) \\
V_2(3) &= \max[5+14,\ 5+13,\ 6+10,\ 8+8] = 19 & (x^* = 0).
\end{aligned}$$

Finally, the grade gain we are interested in

$$V_1(3) = \max[3+19,\ 5+18,\ 6+15,\ 7+13] = 23 \qquad (x^* = 1).$$

Therefore, the maximum grade gain is 23, and the optimal allocation is

2 days for course 1, 1 day for course 2, 3 days for course 3, 1 day for course 4.

2. Let stage be the day. Let 0 = *present*, 1 = Monday, 2 = Tuesday, 3 = Wednesday, and 4 = Thursday. Define $V_n(x_n)$ to be maximum profit if at day n he is in city x_n. Clearly

$$\begin{aligned} V_3(B) &= 16 - 5 = 11 \\ V_3(C) &= 17 - 2 = 15 \\ V_3(I) &= 12 - 0 = 12. \end{aligned}$$

The DPE for this problem is that

$$V_j(x) = \max_{y \in \{B,C,I\}} [\text{profit for staying in city } x - \text{travel cost from city } x \text{ to city } y + V_{j+1}(y)]$$

for $j = 1, 2$. It follows that

$$\begin{aligned} V_2(B) &= \max[16 - 0 + 11,\ 16 - 7 + 15,\ 16 - 5 + 12] = 27 \qquad (y^* = B) \\ V_2(C) &= \max[17 - 7 + 11,\ 17 - 0 + 15,\ 17 - 2 + 12] = 32 \qquad (y^* = C) \\ V_2(I) &= \max[12 - 5 + 11,\ 12 - 2 + 15,\ 12 - 0 + 12] = 25 \qquad (y^* = C) \end{aligned}$$

and

$$\begin{aligned} V_1(B) &= \max[16 - 0 + 27,\ 16 - 7 + 32,\ 16 - 5 + 25] = 43 \qquad (y^* = B) \\ V_1(C) &= \max[17 - 7 + 27,\ 17 - 0 + 32,\ 17 - 2 + 25] = 49 \qquad (y^* = C) \\ V_1(I) &= \max[12 - 5 + 27,\ 12 - 2 + 32,\ 12 - 0 + 25] = 42 \qquad (y^* = C). \end{aligned}$$

Therefore, the maximum profit is

$$V_0(B) = \max[-0 + 43,\ -7 + 49,\ -5 + 42] = 43 \qquad (y^* = B).$$

In other words, the optimal travel route is

$$B \to B \to B \to B \to I,$$

and the optimal profit is 43. □

3. (a) For this system, stage is year, $1, \cdots, N$. State s_n at stage n is the wealth we have at the beginning of the n-th year. The dynamics of the system is

$$s_{n+1} = (1+k)(s_n - c_n);$$

here c_n is the consumption during year n. For convenience, we write $K = 1 + k$. Therefore, we have

$$s_{n+1} = K(s_n - c_n).$$

The optimization problem is to choose a consumption sequence $\{c_n\}$ so as to

$$\max_{\{c_n\}} \sum_{n=1}^{N} (c_n)^a$$

under the constraints that $0 \le c_n \le s_n$ for every n. Define

$$f_j(w) = \max_{\{c_n\}} \sum_{n=j}^{N} (c_n)^a$$

as the maximum happiness we can get during year $j, j+1, \cdots, N$ if at the beginning of year n we have wealth w. The DPE can then be written as

$$f_j(w) = \max_{0 \le c \le w} \left[c^a + f_{j+1}(K(w-c)) \right],$$

with terminal condition $f_N(w) = w^a$ for obvious reason.

(b) Set $b_N = 1$ and $a_N = 1$. We have $f_N(w) = b_N w^a$ and $c_N^* = a_N w$. By induction, assume $f_{n+1}(w) = b_{n+1} w^a$ and $c_{n+1}^* = a_{n+1} w$. By DPE we have

$$f_n(w) = \max_{0 \le c \le w} \left[c^a + f_{n+1}(K(w-c)) \right] = \max_{0 \le c \le w} \left[c^a + b_{n+1} K^a (w-c)^a \right].$$

Write $c = tw$. The max becomes

$$f_n(w) = \max_{0 \le t \le 1} \left[t^a w^a + b_{n+1} K^a (1-t)^a w^a \right] = w^a \cdot \max_{0 \le t \le 1} \left[t^a + b_{n+1} K^a (1-t)^a \right].$$

Set

$$b_n \doteq \max_{0 \le t \le 1} \left[t^a + b_{n+1} K^a (1-t)^a \right],$$

and a_n the maximizing $t^* \in [0,1]$. We have

$$f_n(w) = b_n w^a, \qquad c_n^* = a_n w.$$

Therefore, it is optimal to consume a fixed proportion (the proportion could change from year to year) of the wealth every year.