AM 121: Homework # 6 (solution)

10-3.7 Electronic system.

This is a resuorce allocation problem with multiplicative costs. We want to

$$\text{Maximize} \quad \prod_{n=1}^{4} r_n(x_n)$$

such that

$$\sum_{n=1}^{4} g_n(x_n) \leq 10 \quad \text{(hundred)}.$$

and $x_n \geq 1$ are integers.

The variable x_n stand for the number of parallel units for component n, $g_n(x_n)$ is the cost (in hundreds) of x_n parallel units for component n; and $r_n(x_n)$ is the probability of functioning if x_n parallel units are installed.

Let

$$V_j(w) \doteq \prod_{n=j}^{4} r_n(x_n)$$

such that

$$\sum_{n=j}^{4} g_n(x_n) \leq w \quad \text{(hundred)}.$$

and $x_n \geq 1$ are integers.

The DPE is

$$V_j(w) = \max_{1 \leq x \leq w} \left[r_j(x) \cdot V_{j+1}\left(w - g_j(x_j)\right) \right].$$

We have

$$V_4(0) = V_4(1) = 0, \quad V_4(2) = 0.5, \quad V_4(3) = 0.7, \quad V_4(4) = V_4(5) = \cdots = V_4(10) = 0.9$$

and

$$V_3(0) = V_3(1) = V_3(2) = 0, \quad V_3(3) = 0.7 \cdot 0.5 = 0.35, \quad V_3(4) = 0.7 \cdot 0.7 = 0.49$$

$$V_3(5) = \max\left[0.7 \cdot 0.9, 0.8 \cdot 0.2\right] = 0.63, \quad V_3(6) = \max\left[0.7 \cdot 0.9, 0.8 \cdot 0.7, 0.9 \cdot 0.5\right] = 0.63;$$

$$V_3(7) = \max\left[0.7 \cdot 0.9, 0.8 \cdot 0.9, 0.9 \cdot 0.7\right] = 0.63,$$

$$V_3(8) = V_3(9) = V_3(10) = \max\left[0.7 \cdot 0.9, 0.8 \cdot 0.9, 0.9 \cdot 0.9\right] = 0.81;$$

Similarly, one can compute that

$$V_2(1) = V_2(2) = V_2(3) = V_2(4) = 0,$$

$V_2(5) = 0.21,\ \ V_2(6) = 0.294,\ \ V_2(7) = 0.378,\ \ V_2(8) = 0.378,\ \ V_2(9) = 0.441,\ \ V_2(10) = 0.504.$

The quantity we are interested in is

$$V_1(10) = \max[0.5 \cdot 0.441,\ 0.6 \cdot 0.378,\ 0.8 \cdot 0.378] = 0.3024.$$

Thus the optimal distribution is 3 units for component 1, 1 unit for component 2, 1 unit for component 3, and 3 unit for component 4. The maximal probability is 0.3024.

11.1-2(a) Household chores
let $x_1 = 1$ if Eve do Marketing and $x_1 = 0$ otherwise; $x_2 = 1$ if Eve do Cooking and $x_2 = 0$ otherwise; $x_3 = 1$ if Eve do Dishwashing and $x_3 = 0$ otherwise; $x_4 = 1$ if Eve do Laundry and $x_4 = 0$ otherwise.

The objective is:

Minimize $z = 4.5x_1 + 7.8x_2 + 3.6x_3 + 2.9x_4 + 4.9(1 - x_1) + 7.2(1 - x_2) + 4.3(1 - x_3) + 3.1(1 - x_4)$
with constraint:
$x_1 + x_2 + x_3 + x_4 = 2$ and $x_i = 0$ or 1.

11.3-1(a) Progressive company
let $y_i, i = 1...4$ be the binary variable which is one when i^{th} product is on line, and $y_5 = 1$ if the second constraint in 3 is used and 0 otherwise. So the objective is :
Maximize

$$\begin{aligned} Z &= x_1 70 + x_2 60 + x_3 90 + x_4 80 \\ &\quad - y_1 50000 - y_2 40000 - y_3 70000 - y_4 60000 \end{aligned}$$

under constrain:
$x_1 \leq M_1 y_1, x_2 \leq M_2 y_2, x_3 \leq M_3 y_3, x_4 \leq M_4 y_4$
$y_1 + y_2 + y_3 + y_4 \leq 2$
$y_1 + y_2 \geq y_3$
$y_1 + y_2 \geq y_4$
$5x_1 + 3x_2 + 6x_3 + 4x_4 \leq 6000 + M_5 y_5$
$4x_1 + 6x_2 + 3x_3 + 5x_4 \leq 6000 + M_5(1 - y_5)$

11.3-4(a) Toys production:

Let $v = 1$ if factory 1 is used and $v = 0$ if factory 2 is used. Let $y_1 = 1$ if toy 1 are produced and $y_1 = 0$ otherwise. Similarly, Let $y_2 = 1$ if toy 2 are produced and $y_2 = 0$ otherwise. Let x_1, x_2 be the number of toy 1 and 2 produced.

We want to

$$\text{Maximize} \quad Z = 10x_1 + 15x_2 - 50000y_1 - 80000y_2$$

such that

$$\begin{aligned} x_1 &\le M_1 y_1 \\ x_2 &\le M_2 y_2 \\ \frac{1}{50}x_1 + \frac{1}{40}x_2 &\le 500 + M_3 v \\ \frac{1}{40}x_1 + \frac{1}{25}x_2 &\le 700 + M_3(1-v) \end{aligned}$$

and $x_1, x_2 \ge 0$ integers, y_1, y_2, v binary.

- The dynamic programming problem.

1. For this system, stage is year, $1, \cdots, N$. State s_n at stage n is the wealth we have at the beginning of the n-th year. The dynamics of the system is

$$s_{n+1} = (1+k)(s_n - c_n);$$

here c_n is the consumption during year n. For convenience, we write $K = 1 + k$. Therefore, we have

$$s_{n+1} = K(s_n - c_n).$$

The optimization problem is to choose a consumption sequence $\{c_n\}$ so as to

$$\max_{\{c_n\}} \sum_{n=1}^{N} (c_n)^a$$

under the constraints that $0 \le c_n \le s_n$ for every n. Define

$$f_j(w) = \max_{\{c_n\}} \sum_{n=j}^{N} (c_n)^a$$

as the maximum happiness we can get during year $j, j+1, \cdots, N$ if at the beginning of year n we have wealth w. The DPE can then be written as

$$f_j(w) = \max_{0 \le c \le w} \left[c^a + f_{j+1}(K(w-c)) \right],$$

with terminal condition $f_N(w) = w^a$ for obvious reason.

2. Set $b_N = 1$ and $a_N = 1$. We have $f_N(w) = b_N w^a$ and $c_N^* = a_N w$. By induction, assume $f_{n+1}(w) = b_{n+1} w^a$ and $c_{n+1}^* = a_{n+1} w$. By DPE we have

$$f_n(w) = \max_{0 \le c \le w} \left[c^a + f_{n+1}(K(w-c)) \right] = \max_{0 \le c \le w} \left[c^a + b_{n+1} K^a (w-c)^a \right].$$

Write $c = tw$. The max becomes

$$f_n(w) = \max_{0 \le t \le 1} \left[t^a w^a + b_{n+1} K^a (1-t)^a w^a \right] = w^a \cdot \max_{0 \le t \le 1} \left[t^a + b_{n+1} K^a (1-t)^a \right].$$

Set

$$b_n \doteq \max_{0 \le t \le 1} \left[t^a + b_{n+1} K^a (1-t)^a \right],$$

and a_n the maximizing $t^* \in [0,1]$. We have

$$f_n(w) = b_n w^a, \qquad c_n^* = a_n w.$$

Therefore, it is optimal to consume a fixed proportion (the proportion could change from year to year) of the wealth every year.