AM 121: Homework # 5 (solution)

10.3-2. One can regard this as a general resource allocation problem (note that different formulations can be applied to this problem). Let $1+x_n$ be the days assigned to course n, for $n = 1, 2, 3, 4$. Then we face the allocation problem

$$\text{Maximize} \quad \sum_{n=1}^{4} r_n(1+x_n)$$

such that

$$\sum_{n=1}^{4} x_n \leq 3, \qquad x_n \text{ is non-nagative integer.}$$

Here r_n is the grade gain for course n. Let

$$V_j(w) \doteq \max_{\{x_n\}} \sum_{n=j}^{4} r_n(1+x_n)$$

such that

$$\sum_{n=j}^{4} x_n \leq w.$$

We have

$$V_4(0) = r_4(1) = 6, \quad V_4(1) = r_4(2) = 7, \quad V_4(2) = r_4(3) = 9, \quad V_4(3) = r_4(4) = 9.$$

The DPE is

$$V_j(w) = \max_{\{0 \leq x \leq w\}} \left[r_j(1+x) + V_{j+1}(w-x) \right].$$

We can recursively compute

$$\begin{aligned}
V_3(0) &= 2+6 = 8 && (x^* = 0)\\
V_3(1) &= \max[2+7,\ 4+6] = 10 && (x^* = 1)\\
V_3(2) &= \max[2+9,\ 4+7,\ 7+6] = 13 && (x^* = 2)\\
V_3(3) &= \max[2+9,\ 4+9,\ 7+7,\ 8+6] = 14 && (x^* = 2, 3)
\end{aligned}$$

and

$$\begin{aligned}
V_2(0) &= 5+8 = 13 && (x^* = 0)\\
V_2(1) &= \max[5+10,\ 6+8] = 15 && (x^* = 0)\\
V_2(2) &= \max[5+13,\ 5+10,\ 6+8] = 18 && (x^* = 0)\\
V_2(3) &= \max[5+14,\ 5+13,\ 6+10,\ 8+8] = 19 && (x^* = 0).
\end{aligned}$$

Finally, the grade gain we are interested in

$$V_1(3) = \max[3+19,\ 5+18,\ 6+15,\ 7+13] = 23 \qquad (x^* = 1).$$

Therefore, the maximum grade gain is 23, and the optimal allocation is

2 days for course 1, 1 day for course 2, 3 days for course 3, 1 day for course 4.

2. Shortest path
The table is:

n	SN	CCUN	total distance	NNN	Minimum distance	last connection
1	O	A	4	A	4	OA
2,3	O	C	5	C	5	OC
	A	B	5	B	5	AB
	A	D	11			
4	B	E	9	E	9	BE
	C	E	10			
	A	D	11			
5	B	D	10	D	10	BD
	E	D	10	D	10	ED
6	D	T	16	T	16	DT
	E	T	17			

SN stands for “Solved Nodes directly connected to unsolved nodes”, CCUN for “closest connected unsolved nodes”,NNN for “nth nearest node”.
Thus we see the shortest routine is $O \to A \to B \to E \to D \to T$, or $O \to A \to B \to D \to T$, with the shortest distance 16.

3. Let stage be the day. Let 0 = *present*, 1 = Monday, 2 = Tuesday, 3 = Wednesday, and 4 = Thursday. Define $V_n(x_n)$ to be maximum profit if at day n he is in city x_n. Clearly

$$\begin{aligned} V_3(B) &= 16-5=11 \\ V_3(C) &= 17-2=15 \\ V_3(I) &= 12-0=12. \end{aligned}$$

The DPE for this problem is that

$$V_j(x) = \max_{y\in\{B,C,I\}} [\text{profit for staying in city } x - \text{travel cost from city } x \text{ to city } y + V_{j+1}(y)]$$

for $j = 1, 2$. It follows that

$$\begin{aligned} V_2(B) &= \max[16-0+11,\ 16-7+15,\ 16-5+12] = 27 && (y^* = B) \\ V_2(C) &= \max[17-7+11,\ 17-0+15,\ 17-2+12] = 32 && (y^* = C) \\ V_2(I) &= \max[12-5+11,\ 12-2+15,\ 12-0+12] = 25 && (y^* = C) \end{aligned}$$

and

$$\begin{aligned} V_1(B) &= \max[16-0+27,\ 16-7+32,\ 16-5+25] = 43 && (y^* = B) \\ V_1(C) &= \max[17-7+27,\ 17-0+32,\ 17-2+25] = 49 && (y^* = C) \\ V_1(I) &= \max[12-5+27,\ 12-2+32,\ 12-0+25] = 42 && (y^* = C). \end{aligned}$$

Therefore, the maximum profit is

$$V_0(B) = \max[-0+43,\ -7+49,\ -5+42] = 43 \qquad (y^* = B).$$

In other words, the optimal travel route is

$$B \to B \to B \to B \to I,$$

and the optimal profit is 43. □