Solns to 4^{th} Homework

6.3-5 (a) primal and dual

primal LP problem	dual LP problem
max $Z = 2x_1 + 4x_2$	min $W = y$
$x_1 - x_2 \leq 1$	$y \geq 0$
$x_1 \geq 0$	$y \geq 2$
$x_2 \geq 0$	$-y \geq -4$

We can see the optimal of dual LP problem is 2 when y=2.

(b) complete slackness

By Complete slackness theorem, we know optimal x and dual optimal y satisfy those equations:

$$(b - Ax)^T y = 0 \text{ and } (A^T y - c)^T x = 0.$$

Replace these vectors by what they are in this problem, we can get:

$$x_1 - x_2 = 1$$
$$(0, 2)(x_1, x_2)^T = 0$$

it's easy to find the solution of this group equations, that is $x_1 = 1, x_2 = 0$.

(c) reconstraint

Now the dual LP problem is:

Minmize $w = y$ subject to $y \geq 0, y \geq c_1, -y \geq -4$

we can see when $c_1 > 4$, there is no feasible solution for dual problem. Notice that changes in objective function don't affect the feasible region. Thus the primal problem is unbounded.

6.7-3 change

The final simplex tableau is

Basic Variable	Row	Z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
Z	0	1	0	0	3/2	0	3/2	1/2	25
x_4	1	0	0	0	1	1	-1	-2	10
x_1	2	0	1	0	1/2	0	1/2	1/2	15
x_2	3	0	0	1	-3/2	0	-1/2	1/2	5

(a) change the right hand side
The only change that happens is in the last column. The problem is whether or not the optimal solution now is still feasible. Let' s check out $B^{-1} \cdot \bar{b}$

$$B^{-1} \cdot \bar{b} = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 70 \\ 20 \\ 10 \end{pmatrix} = \begin{pmatrix} 30 \\ 15 \\ -5 \end{pmatrix}$$

Which tells us that the optimal solution now is not feasible.

(c) change the coefficients of x_3
Noticing x_3 is not a basic variable, thus this change only affects the x_3 column. The coefficient of x_3 in row(0) will change into $C_B^T B^{-1} \bar{A}_j - \bar{c}_j$, the column except row(0) will be $B^{-1}\bar{A}_j$. Check out these values by replacing vectors into it.

$$C_B^T B^{-1} A_j - \bar{c}_j = \begin{pmatrix} 0 & 3/2 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} - 2 = -3/2$$

So now the solution is not optimal any more. since

$$B^{-1}\bar{A}_j = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ -1/2 \\ -3/2 \end{pmatrix}$$

The new simplex table will be

Basic Variable	Row	Z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
Z	0	1	0	0	-3/2	0	3/2	1/2	25
x_4	1	0	0	0	6	1	-1	-2	10
x_1	2	0	1	0	-1/2	0	1/2	1/2	15
x_2	3	0	0	1	-3/2	0	-1/2	1/2	5

iteration(0)

Basic Variable	Row	Z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
Z	0	1	0	0	0	1/4	5/4	0	55/2
x_3	1	0	0	0	1	1/6	-1/6	-1/3	5/3
x_1	2	0	1	0	0	1/12	5/12	1/3	95/6
x_2	3	0	0	1	0	1/4	-3/4	0	15/2

see there is no negative number in first row, thus this is the optimal solution.

(d) change the objective
Check out the new simplex form. Just reconstruct the first row

$$C_B^T B^{-1} A - c^T = \begin{pmatrix} 0 & 3 & -2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1/2 \\ 0 & 1 & -3/2 \end{pmatrix} - \begin{pmatrix} 3 & -2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 3/2 \end{pmatrix}$$

$$C_B^T B^{-1} = \begin{pmatrix} 0 & 3 & -2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 0 & 5/2 & 1/2 \end{pmatrix}$$

$C_B^T B^{-1} b = \begin{pmatrix} 0 & 5/2 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 60 & 10 & 20 \end{pmatrix}^T = 35$

Now we see the first row is not negative ,so it's still optimal after changing some coefficients of basic variable.

(f) introduce a new variable
just reconstruct the simplex form, taking the new variable as a non-basic variable.

$$C_B^T B^{-1} A_8 - c_8 = \begin{pmatrix} 0 & 3/2 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} + 1 = 7/2$$

which ,positive, shows that the solution is still optimal.