AM 121: Homework # 3 (solution)

1. (a) primal and dual

primal LP problem	dual LP problem
$maxZ = x_1 + 3x_2$	$minW = -6y_1 + 2y_2$
$-3x_1 - x_2 \leq -6$	$y_1 \geq 0$
$2x_1 - x_2 = 2$	*no sign constraint on* y_2
$x_1 \geq 0$	$-3y_1 + 2y_2 \geq 1$
$x_2 \geq 0$	$-y_1 - y_2 \geq 3$

(b)primal and dual

primal LP problem	dual LP problem
$maxZ = x_1 + 3x_2$	$minW = -6y_1 + 2y_2 - 2y_3$
$-3x_1 - x_2 \leq -6$	$y_1 \geq 0$
$2x_1 - x_2 \leq 2$	$y_2 \geq 0$
$-2x_1 + x_2 \leq -2$	$y_3 \geq 0$
$x_1 \geq 0$	$-3y_1 + 2y_2 - 2y_3 \geq 1$
$x_2 \geq 0$	$-y_1 - y_2 + y_3 \geq 3$

(c) Equivalence For dual(a),as there is no constraint on y_2, we can let $y_2 = u - v$, with $u \geq 0, v \geq 0$, then we have:
objective min $w = -6y_1 + 2u - 2v$ with $y_1 \geq 0$,$u \geq 0$, $v \geq 0$ under the constraints: $-3y_1 + 2u - 2v \geq 1$ and $-y_1 - u + v \geq 3$ which has the same form as in dual(b).

(d) Unbounded primal and infeasible dual
Just plot the constraint picture of dual LP problem, we can check it out there is no feasible region at al. Of course you can show it from the lemma.

2. Find optimal value use this table in book

BV	Row	Z	x	s	RHS
Z	(0)	1	$C_B^T B^{-1}A - c^T$	$C_B^T B^{-1}$	$C_B^T B^{-1}b$
x_B	:	0	$B^{-1}A$	B^{-1}	$B^{-1}b$

Now you see $C_B^T B^{-1} = (0.4, 1.4)$ and $b = (0.5, 0.5)$, thus optimal of the LP problem is $0.4 \times 0.5 + 1.4 \times 0.5 = 0.9$.

3. Reconstruct LP problem

(a) Find which is optimal
From above table, we know the first row now is $C_B^T B^{-1}A - c^T$, replace these vectors into it, you will see the first row is (4.5,-2,0,0,2.5,0) when you take the first B^{-1} into it, which is not optimal because of the negative number.
And the result with pluging the second B^{-1} into it is (2,8,0,5,0,0), which is the optimal case because it's not negative.

(b) Recover the LP simplex table using above form. You can get

BV	Row	Z	x_1	x_2	x_3	s_1	s_2	s_3	RHS
Z	0	1	2	8	0	5	0	0	150
x_3	1	0	1	2	1	1	0	0	30
s_2	2	0	1	-4	0	-2	1	0	0
s_3	3	0	1	4	0	0	0	1	20

See in the optimal case s_2 takes a value as 0, thus this LP problem is degenerate.

(c) Constraint
We see now b=(R,60,20) and notice that $C_B^T B^{-1}A - c^T$, $B^{-1}A$ will not change when b changes. Thus in order to keep optimal, we only need to keep $B^{-1}b \geq 0$, to keep the solution feasible. Replace these vectors into this constraint, we have:
$-2R + 60 \geq 0$, so when $0 \leq R \leq 30$, the solution is still optimal.

4. Problem 6.3.5
(a)

$$\text{Minimize } v = y_1$$

$$\begin{aligned} s.t \;\; y_1 &\geq 2 \\ -y_1 &\geq -4 \\ y_1 &\geq 0 \end{aligned}$$

So the optimal solution is $v = y_1 = 2$.

(b) $y_1 = 2$ is the optimal basic feasible solution for the dual. By the complementary slackness, $x = (x_1, x_2)$ is optimal if and if $(1 - x_1 + x_2)y_1 = 0$, $(y_1 - 2)x_1 = 0$, $(-y_1 + 4)x_2 = 0$ all hold. Given $y_1 = 2$, solve for x_1 and x_2. The optimal x is (1,0).

(c) For $c_1 > 4$,the dual will have no feasible solution. Therefore, the primal objective function will be unbounded.

5. Probelm 6.7.3
(a) change b.
$b = \begin{pmatrix} 20 \\ 30 \end{pmatrix}$ $B^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$
$B^{-1} \cdot b = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 20 \\ 30 \end{pmatrix} = \begin{pmatrix} -10 \\ 30 \end{pmatrix}$
Since RHS can't have negative entry, the current basic variables are not feasible.

(b) change coefficients of x_3
$c_3 = -2$ $A_3 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$C_B^T B^{-1} = (0,2) \quad B^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

so $C_B^T B^{-1} A_3 - c_3 = (0,2) \begin{pmatrix} 3 \\ -2 \end{pmatrix} - (-2) = -2$

$B^{-1} A_3 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

Since coefficient of x_3 in Row(0) is negative, the current basic variables are not optimal. Reoptimize it as follows.

BV	Row	Z	x_1	x_2	x_3	x_4	x_5	RHS	$Ratio$
Z	0	1	0	1	−2	0	2	20	
x_1	1	0	0	−1	5	1	−1	20	4
x_4	2	0	1	4	−2	0	1	10	NA

BV	Row	Z	x_1	x_2	x_3	x_4	x_5	RHS	$Ratio$
Z	0	1	0	3/5	0	2/5	8/5	28	
x_3	1	0	0	−1/5	1	1/5	−1/5	4	
x_4	2	0	1	18/5	0	2/5	3/5	18	

(d) Add x_6

$c_6 = -3 \quad A_6 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$C_B^T B^{-1} = (0,2)$

so $C_B^T B^{-1} A_6 - c_6 = (0,2) \begin{pmatrix} 1 \\ 2 \end{pmatrix} - (-3) = 7$

Since coefficient in Row(0) for x_6 is positive, the current basic variables are still optimal.

(e) change objective function, i.e change c. Note: Only Row(0) will change, the other rows stay unchanged.

Basic variables =(x_4, x_1), $c_B^T = (0,1)$, $c^T = (1,5,-2)$

$B^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad B^{-1} A = \begin{pmatrix} 0 & -1 & 5 \\ 1 & 4 & -1 \end{pmatrix} \quad B^{-1} b = \begin{pmatrix} 20 \\ 10 \end{pmatrix}$

so $C_B^T B^{-1} A - c^T = (0,1) \begin{pmatrix} 0 & -1 & 5 \\ 1 & 4 & -1 \end{pmatrix} - (1,5,-2) = (0,-1,1)$

$C_B^T B^{-1} = (0,1) \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = (0,1)$

$c_B^T B^{-1} b = (0,1) \begin{pmatrix} 20 \\ 10 \end{pmatrix} = 10$

Since there is negative coefficient in Row(0), the current basic variables are not optimal. Reoptimize it as follows

BV	Row	Z	x_1	x_2	x_3	x_4	x_5	RHS	$Ratio$
Z	0	1	0	−1	1	0	1	10	
x_4	1	0	0	−1	5	1	−1	20	
x_1	2	0	1	4	−1	0	1	10	5/2

BV	Row	Z	x_1	x_2	x_3	x_4	x_5	RHS	$Ratio$
Z	0	1	1/4	0	3/4	0	5/4	25/2	
x_4	1	0	1/4	0	19/4	1	$-3/4$	45/2	
x_2	2	0	1/4	1	$-1/4$	0	1/4	5/2	