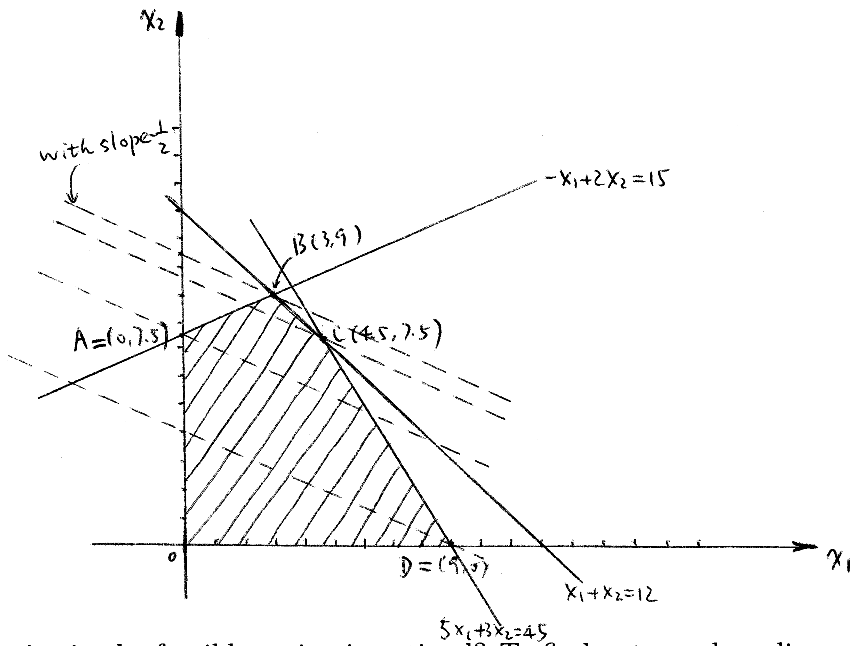
Solutions to Homework #1

21 Sep 2002

1 3.1-5.

Solution: By the **graphical method**, every point in the shaded region below stands for an allocation (x_1, x_2) that satisfies all the constraints.



Which point in the feasible region is optimal? To find out, we draw lines of constant

$$10x_1 + 20x_2 = \text{constant}.$$

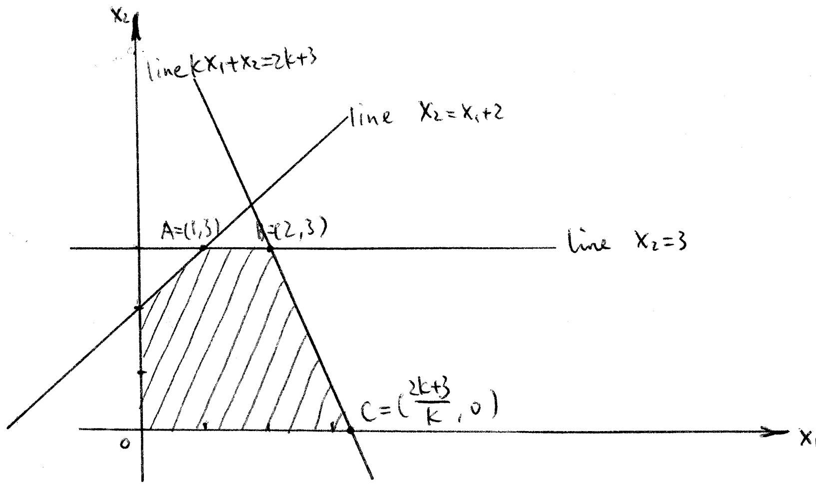
These are the lines with slope $-1/2$, hence they are parallel to each other. All points (x_1, x_2) on each of these lines give the same objective function value. Draw the line of slope $-1/2$ through point A, B, C, D respectively. This gives four parallel lines among which point B gives the greatest value. In other words, **the point B is an optimal solution**; indeed, in this case point B is the *unique* optimal solution. It is not difficult to see that $B = (3, 9)$, and the optimal objective function value is

$$Z = 10 \times x_1 + 20 \times x_2 = 10 \times 3 + 20 \times 9 = 210.$$

As a check, note $A = (0, 7.5)$ and $C = (4.5, 7.5)$, and they make Z equal 150 and 195 respectively. The are both suboptimal.

2 3.1-13.

Solution: Note that point $B : (x_1, x_2) = (2, 3)$ is always on the line $kx_1 + x_2 = 2k + 3$ where $k \geq 0$, so the only situation that point B could be the optimal solution is drawn as following roughly:



If point B is actually optimal, there is no better *neighboring* point, i.e. point $A = (1, 3), C = (\frac{2k+3}{k}, 0)$ are both suboptimal, which means

$Z(B) = 2 + 2 \times 3 = 8 > Z(A) = 1 + 2 \times 3 = 7$(always true!)

$Z(B) = 8 > Z(C) = \frac{2k+3}{k} + 2 \times 0 = \frac{2k+3}{k} \Rightarrow k > 1/2$.

Therefore, as long as $k > 1/2$, point $B : (x_1, x_2) = (2, 3)$ is optimal.

3 3.2-2.

Solution:

(a) **TRUE**. When we study 'the intuitive idea behind *Simplex Algorithm*' in the notes, we know the fact that 'a vertex is an optimal solution if there is no better *neighboring vertex*'. Here the objective function value of point $(3, 3)$ beats those of point $(0, 2)$ and point $(6, 3)$, so $(3, 3)$ must be an optimal solution.

(b) **TRUE**. If there are multiple optimal solutions, then at least two must be adjacent corner-point feasible solutions. In our case, either $(0, 2)$ or $(6, 3)$ must be an optimal solution, but can't both!!

(c) **FALSE**. Just check the case that if we want to maximize $Z = -x_1 - x_2$ in the feasible region, point $(0, 0)$ **IS** the optimal solution (all the other points in the feasible region correspond to negative objective function values).

4 3.4-12(a).

Solution: Note that $M1, M2$ produce 40 tons and 60 tons iron ores respectively and P needs 100 tons every month, which means total 100 tons produced by $M1, M2$ will be shipped to P through $S1$ and $S2$ with no inventory at $S1$ and $S2$. Instead of using messy notation as done in the textbook, we suppose:

from $M1$ to $S1$ we transport x_1 tons,

from $M2$ to $S2$ we transport x_2 tons,

so

from $M1$ to $S2$ we transport $40 - x_1$ tons,

from $M2$ to $S1$ we transport $60 - x_2$ tons,

from $S1$ to P we transport $x_1 + 60 - x_2$ tons,

from $S2$ to P we transport $40 - x_1 + x_2$ tons.

Therefore, this problem reduces to the following linear programming:

$$\begin{aligned} \max Z &= 2000x_1 + 1700(40 - x_1) + 1100x_2 + 1600(60 - x_2) + 400(x_1 + 60 - x_2) + 800(40 - x_1 + x_2) \\ &= -100x_1 - 100x_2 + 22000 \end{aligned}$$

subject to the following constraints:

$$\begin{aligned} x_1 &\leq 30 \\ x_2 &\leq 50 \\ 40 - x_1 &\leq 30 \\ 60 - x_2 &\leq 50 \\ x_1 + 60 - x_2 &\leq 70 \\ 40 - x_1 + x_2 &\leq 70 \\ x_1 &\geq 0 \\ x_2 &\geq 0. \end{aligned}$$

5 Extra problem.

Solution: Notice that the following two programmings are equivalent:

$$\text{Minimize } |3x_1 + 4x_2 - 7|$$

and

$$\begin{aligned} &\text{Minimize } z_1 \\ &\text{s.t. } |3x_1 + 4x_2 - 7| \leq z_1. \end{aligned}$$

The constraints is namely

$$\begin{aligned} 3x_1 + 4x_2 - 7 &\leq z_1 \\ 3x_1 + 4x_2 - 7 &\geq -z_1. \end{aligned}$$

Why this is true? By noticing that $|3x_1 + 4x_2 - 7|$ is the lower bound of z_1, actually the minimization of $|3x_1 + 4x_2 - 7|$ is the same as the minimization of z_1.
Therefore, the original problem can be stated as the following linear programming:

$$\begin{aligned} \text{Minimize } & z_1 + 2z_2 + 8z_3 \\ \text{s.t. } 3x_1 + 4x_2 - z_1 &\leq 7 \\ 3x_1 + 4x_2 + z_1 &\geq 7 \\ 2x_1 + 3x_2 - z_2 &\leq 5 \\ 2x_1 + 3x_2 + z_2 &\geq 5 \\ -x_1 + 4x_2 - z_3 &\leq 9 \\ -x_1 + 4x_2 + z_3 &\geq 9. \end{aligned}$$