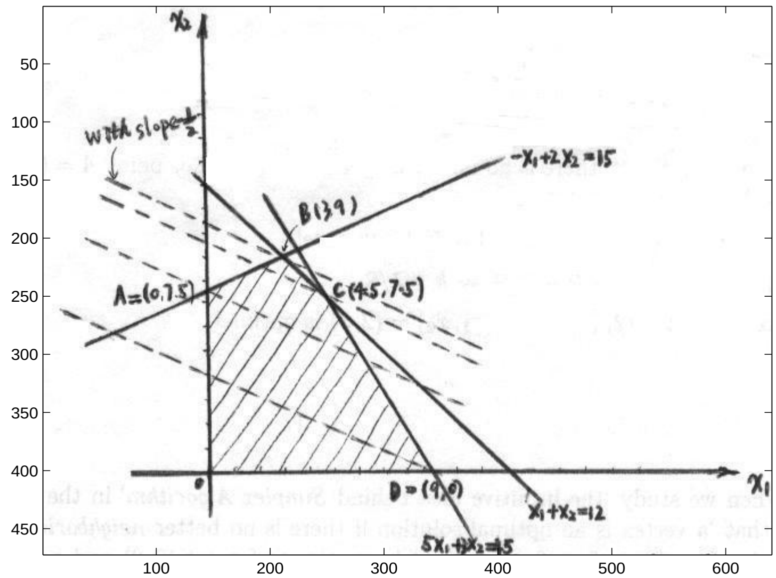
AM 121: Homework # 1 (solution)

1. **3.1-5**

   Solution: By the graphical method, every point in the shaded region below stands for an allocation (x_1, x_2) that satisfies all the constraints.



   Which point in the feasible region is optimal? To find out, we draw lines of constant

$$10x_1 + 20x_2 = \text{constant}$$

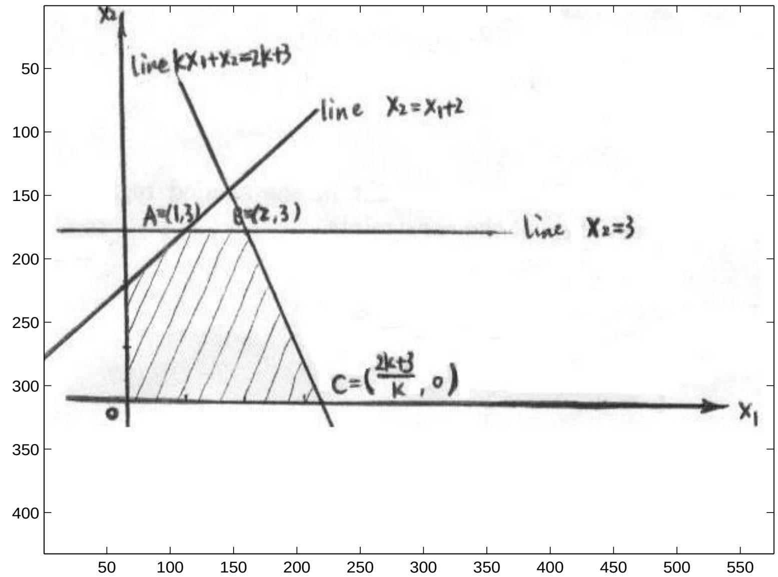
   These are the lines with slope -1/2, hence they are parallel to each other. All points (x_1, x_2) on each of these lines give the same objective function value. Draw the line of slope -1/2 through point A, B, C, D respectively. This gives four parallel lines among which point B gives the greatest value. In other words, **the point B is an optimal solution**; indeed, in this case point B is the *unique* optimal solution. It is not difficult to see that $B = (3, 9)$, and the optimal objective function value is

$$Z = 10x_1 + 20x_2 = 10 \times 3 + 20 \times 9 = 210$$

   As a check, note $A = (0, 7.5)$ and $C = (4.5, 7.5)$, and they make Z equal 150 and 195 respectively. They are both suboptimal.

2. **3.1-12**

   Solution: Note that point $B : (x_1, x_2) = (2, 3)$ is always on the line $kx_1 + x_2 = 2k + 3$ where $k \geq 0$, so the only situation point B could be the optimal solution is drawn as follows roughly.



If point B is actually optimal, there is no better neighboring point, i.e point $A = (1,3)$,$C = (\frac{2k+3}{k}, 0)$ are both suboptimal, which means

$$
\begin{aligned}
Z(B) &= 2 + 2 \times 3 = 8 > Z(A) = 1 + 2 \times 3 = 7 \ (\textit{always true!}) \\
Z(B) &= 8 > Z(C) = \frac{2k+3}{k} + 2 \times 0 = \frac{2k+3}{k} \Rightarrow k > 1/2.
\end{aligned}
$$

Therefore, as long as $k > 1/2$, point $B : (x_1, x_2) = (2, 3)$ is optimal.

3. **3.3-2**

   Solution:

   (a) **TRUE.** When we study the intuitive idea behind *Simplex Algorithm* in the notes, we know the fact that a vertex is an optimal solution if there is no better neighboring vertex. Here the objective function value of the point (3,3) beats those of point (0,2) and point (6,3), so (3,3) must be an optimal solution.

   (b) **TRUE.** If there are multiple optimal solutions, then at least two must be adjacent cornerpoint feasible solution. In our case, either (0,2) or (6,3) must be an optimal solution, but can't both!!

   (c) **FALSE.** Just check the case that if we want to maximize $Z = -x_1 - x_2$ in the feasible region, point (0,0) is the optimal solution (all the other points in the feasible region correspond to negative objective function values)

4. **3.4-13**

   Solution: Let

x_{1F} = number of tons of cargo type 1 stowed in the front compartment
x_{1C} = number of tons of cargo type 1 stowed in the center compartment
x_{1B} = number of tons of cargo type 1 stowed in the back compartment
x_{2F} = number of tons of cargo type 2 stowed in the front compartment
x_{2C} = number of tons of cargo type 2 stowed in the center compartment
x_{2B} = number of tons of cargo type 2 stowed in the back compartment
x_{3F} = number of tons of cargo type 3 stowed in the front compartment
x_{3C} = number of tons of cargo type 3 stowed in the center compartment
x_{3B} = number of tons of cargo type 3 stowed in the back compartment
x_{4F} = number of tons of cargo type 4 stowed in the front compartment
x_{4C} = number of tons of cargo type 4 stowed in the center compartment
x_{4B} = number of tons of cargo type 4 stowed in the back compartment

Maximize $P = 320x_{1F} + 320x_{1C} + 320x_{1B} + 400x_{2F} + 400x_{2C} + 400x_{2B} + 360x_{3F} + 360x_{3C} + 360x_{3B} + 290x_{4F} + 290x_{4C} + 290x_{4B}$
subject to

$$\begin{aligned}
x_{1F} + x_{2F} + x_{3F} + x_{4F} &\leq 12 \\
x_{1C} + x_{2C} + x_{3C} + x_{4C} &\leq 18 \\
x_{1B} + x_{2B} + x_{3B} + x_{4B} &\leq 10 \\
x_{1F} + x_{1C} + x_{1B} &\leq 20 \\
x_{2F} + x_{2C} + x_{2B} &\leq 16 \\
x_{3F} + x_{3C} + x_{3B} &\leq 25 \\
x_{4F} + x_{4C} + x_{4B} &\leq 13 \\
500x_{1F} + 700x_{2F} + 600x_{3F} + 400x_{4F} &\leq 7000 \\
500x_{1C} + 700x_{2C} + 600x_{3C} + 400x_{4C} &\leq 9000 \\
500x_{1B} + 700x_{2B} + 600x_{3B} + 400x_{4B} &\leq 5000
\end{aligned}$$

$$\begin{aligned}
\frac{1}{12}x_{1F} &+ \frac{1}{12}x_{2F} + \frac{1}{12}x_{3F} + \frac{1}{12}x_{4F} - \frac{1}{18}x_{1C} - \frac{1}{18}x_{2C} - \frac{1}{18}x_{3C} - \frac{1}{18}x_{4C} = 0 \\
\frac{1}{12}x_{1F} &+ \frac{1}{12}x_{2F} + \frac{1}{12}x_{3F} + \frac{1}{12}x_{4F} - \frac{1}{10}x_{1B} - \frac{1}{10}x_{2B} - \frac{1}{10}x_{3B} - \frac{1}{10}x_{4B} = 0
\end{aligned}$$

$$\begin{aligned}
x_{1F} &\geq 0, x_{1C} \geq 0, x_{1B} \geq 0, x_{2F} \geq 0, x_{2C} \geq 0, x_{2B} \geq 0 \\
x_{3F} &\geq 0, x_{3C} \geq 0, x_{3B} \geq 0, x_{4F} \geq 0, x_{4C} \geq 0, x_{4B} \geq 0.
\end{aligned}$$

Extra problem

Solution: The problem is equivalent to

$$\text{Min } Z = y_1 + 2y_2 + 8y_3$$

$$\begin{aligned} s.t\ |3x_1 + 4x_2 - 7| &\leq y_1 \\ |2x_1 + 3x_2 - 5| &\leq y_2 \\ |-x_1 + 4x_2 - 9| &\leq y_3 \end{aligned}$$

Why this is true? By noticing that $|3x_1 + 4x_2 - 7| + 2 \cdot |2x_1 + 3x_2 - 5| + 8 \cdot |-x_1 + 4x_2 - 9|$ is the lower bound of $y_1 + 2y_2 + 8y_3$, but the minimization of the former is the same as the minimization of the latter.

Note $|3x_1 + 4x_2 - 7| \leq y_1$ is equivalent to

$$\begin{aligned} 3x_1 + 4x_2 - 7 &\leq y_1 \\ 3x_1 + 4x_2 - 7 &\geq -y_1 \end{aligned}$$

Therefore, the original problem can be stated as the following linear programming:

$$\text{Minimize } y_1 + 2y_2 + 8y_3$$

$$\begin{aligned} s.t\ \ 3x_1 &+ 4x_2 - y_1 \leq 7 \\ 3x_1 &+ 4x_2 + y_1 \geq 7 \\ 2x_1 &+ 3x_2 - y_2 \leq 5 \\ 2x_1 &+ 3x_2 + y_2 \geq 5 \\ -x_1 &+ 4x_2 - y_3 \leq 9 \\ -x_1 &+ 4x_2 + y_3 \geq 9 \\ y_i &\geq 0, i = 1, 2, 3 \end{aligned}$$