

Game Theory (Part I)

1 Examples

From the previous section, we know that solving a game is as solving an LP. But for a simple game where each player only has two strategies, the solution is easy to obtain without resorting to LP.

Example: Suppose the payoff matrix is

$$B = \begin{bmatrix} 0 & a \\ c & 0 \end{bmatrix}$$

with $a, c > 0$. Compute the value of the game and the saddle point.

Solution: The row player will choose a mixed strategy so as to

$$\max_x \min_{1 \leq j \leq n} \left(\sum_{i=1}^m x_i b_{ij} \right) = \max_x \min (x_1 b_{11} + x_2 b_{21}, x_1 b_{12} + x_2 b_{22}) = \max_x \min (cx_2, ax_1)$$

Since $x_1 + x_2 = 1$, the maximizing x^* is such that $cx_2^* = ax_1^*$ or

$$x_1^* = \frac{c}{a+c}, \quad x_2^* = \frac{a}{a+c}.$$

The value of the game is therefore

$$v = \max_x \min (cx_2, ax_1) = \min (cx_2^*, ax_1^*) = \frac{ac}{a+c}.$$

Similarly, the column player will choose a mixed strategy (exercise!)

$$y_1^* = \frac{a}{a+c}, \quad y_2^* = \frac{c}{a+c}.$$

A bluffing game: Consider the following simple bluffing game. The first thing we both do is putting down a small positive ante a . Then you draw one card from an ordinary deck. After looking at it, you put it face down without showing it to me. We will say black cards are high, and red cards are low.

Here are the rules. After you draw, you may bet or fold. If you fold, you lose the ante a . If you bet, then I may fold or I may call. If I fold, I lose the ante a , whether you draw black and red. If I bet, then you win the amount b if you drew a black card, or I win the amount b if you drew a red card. (The ante a and bet size b are fixed in advance, with $0 < a < b$.)

Your pure strategies: If you draw black, you will certainly bet – there’s no question about that; you will bet and win at least the ante a . The only question is this: Will you bet if you draw red? That would be the *bluff* strategy. If you fold when you draw red, that is the *fold* strategy. (Remember, if you are playing the fold strategy, you will still bet if you draw black).

My pure strategies: You have just bet. What should I do? If I know you only bet on black, I will fold; but if I think you may bluff and bet on red, I may decide to call you. I have two pure strategies: the *call* strategy, in which I will call if you bet; and the *fold* strategy, in which I will fold if you bet.

The payoff matrix for you is as follows:

		I	
		Call	Fold
You	Bluff	0	a
	Fold	$(b - a)/2$	0

Let us explain the four components.

- $b_{11} = 0$: If you draw black, you will bet, I will call, and you will win b . If you draw red, you will bluff, I will call, and you will lose b . Since black and red are equally likely, your *average* payoff is $(b - b)/2 = 0$.
- $b_{12} = a$: Whatever you draw, you will bet, I will fold, and you will win the ante a .
- $b_{21} = (b - a)/2$: If you draw black, you will bet, I will call, and you will win b . If you draw red, you will fold, and you will lose ante a . Your average payoff is $(b - a)/2$.
- $b_{22} = 0$: If you draw black, you will bet, I will fold, and you will win a . If you draw red, you will fold, and you will lose a . Your average payoff is 0.

What are the optimal strategies for the players? If you always play your *bluff* strategy, I will play my *call* strategy; If you always play *fold*, I will play *fold*. So, in pure strategies, I can hold your payoff down to zero.

It follows from the previous example that you have an optimal mixed strategy telling that you should play the bluff strategy with probability $(b - a)/(b + a)$ and the fold strategy with probability $(2a)/(a + b)$. The value of the game (your expected reward) is

$$v = a \frac{b - a}{b + a}.$$

Your optimal bluffing frequency depends on the bet-to-ante ratio $r = b/a$: the bigger r is, the more often you should bluff.

1.1 How to solve the LPs associated with the game

For more complicated payoff matrix B , we need to solve the LP of form:

$$\text{Maximize } Z = x_{m+1}$$

under constraints

$$\begin{array}{rcccccc} -x_1b_{11} & - & x_2b_{21} & - & \cdots & - & x_mb_{m1} & + & x_{m+1} & \leq & 0 \\ -x_1b_{12} & - & x_2b_{22} & - & \cdots & - & x_mb_{m2} & + & x_{m+1} & \leq & 0 \\ & & \vdots & & & & & & & & \\ -x_1b_{1n} & - & x_2b_{2n} & - & \cdots & - & x_mb_{mn} & + & x_{m+1} & \leq & 0 \\ & & x_1 & + & x_2 & + & \cdots & + & x_m & = & 1 \end{array}$$

and $x_1, \dots, x_m \geq 0$, while x_{m+1} has no sign constraints.

One can certainly transform this LP into standard form, add slack and artificial variables, and use simplex algorithm. An easier approach is as follows: (a) Add a constant, say c , to the payoff matrix so that every entry of the matrix is now non-negative. Of course, this will not change the optimal strategy whatsoever, but will raise the value of the game by c ; (b) Now we can assume $b_{ij} \geq 0$. And the original LP is equivalent to the LP in standard form (why?):

$$\text{Maximize } Z = x_{m+1}$$

under constraints

$$\begin{array}{rcccccc} -x_1b_{11} & - & x_2b_{21} & - & \cdots & - & x_mb_{m1} & + & x_{m+1} & \leq & 0 \\ -x_1b_{12} & - & x_2b_{22} & - & \cdots & - & x_mb_{m2} & + & x_{m+1} & \leq & 0 \\ & & \vdots & & & & & & & & \\ -x_1b_{1n} & - & x_2b_{2n} & - & \cdots & - & x_mb_{mn} & + & x_{m+1} & \leq & 0 \\ & & x_1 & + & x_2 & + & \cdots & + & x_m & \leq & 1 \end{array}$$

and $x_1, \dots, x_m \geq 0, x_{m+1} \geq 0$.

We only need to add slack variables to this LP.

Exercise: Show that the dual of this new LP is equivalent to the dual of the original LP under the assumption that $b_{ij} \geq 0, \forall i, j$. This shows that when we obtain the optimal primal solution by simplex algorithm, we also obtain the optimal dual solution (coefficients for slack variables). Note the optimal primal (resp. dual) solution gives the optimal strategy for the row (resp. column) player.

Two Finger Morra: Two players in the game of Two-Finger Morra simultaneously put out either one or two fingers. Each player must also announce the number of fingers that he believes his opponent has put out. If neither or both players successfully guess the number of fingers put out by the opponent, the game is a draw. Otherwise, the player who guesses correctly wins from the other player the sum (in dollars) of the fingers put out by the two players.

The game is a zero-sum game. Let (i, j) be a strategy of putting out i fingers and guessing the opponent has put out j fingers. The payoff matrix is as follows.

		Column player				Row min
		(1,1)	(1,2)	(2,1)	(2,2)	
Row player	(1,1)	0	2	-3	0	-3
	(1,2)	-2	0	0	3	-2
	(2,1)	3	0	0	-4	-4
	(2,2)	0	-3	4	0	-3
Column max		3	2	4	3	

This game does not satisfy the minimax condition; i.e.

$$\max_i \left(\min_j b_{ij} \right) = -2 \neq 2 = \min_j \left(\max_i b_{ij} \right).$$

Therefore there is no pure-strategy saddle point.

However, zero-sum games always have a value when we consider mixed strategies. In this case, the game is symmetric; i.e. $B = -B^T$. Therefore the value of the game is 0. But what is the optimal strategy for each player? This can be solved by forming the equivalent LP.

First we make each entry in the payoff matrix non-negative by adding a constant, say 4. The payoff matrix is now

$$B = \begin{bmatrix} 4 & 6 & 1 & 4 \\ 2 & 4 & 4 & 7 \\ 7 & 4 & 4 & 0 \\ 4 & 1 & 8 & 4 \end{bmatrix}$$

The optimal strategy will not change by this operation, but the value will increase by 4. For the row player is to find a mixed strategy $x = (x_1, \dots, x_4)$ so as to

$$\max_x \min_y \left(\sum_{i=1}^4 \sum_{j=1}^4 x_i y_j b_{ij} \right) = \max_x \left(\sum_{j=1}^4 x_j b_{ij} \right).$$

The corresponding LP is

$$\text{Maximize } Z = x_5$$

under constraints

$$\begin{aligned} -4x_1 - 2x_2 - 7x_3 - 4x_4 + x_5 &\leq 0 \\ -6x_1 - 4x_2 - 4x_3 - x_4 + x_5 &\leq 0 \\ -x_1 - 4x_2 - 4x_3 - 8x_4 + x_5 &\leq 0 \\ -4x_1 - 7x_2 - 4x_4 + x_5 &\leq 0 \\ x_1 + x_2 + x_3 + x_4 &\leq 1 \end{aligned}$$

and $x_1, \dots, x_4 \geq 0, x_5 \geq 0$.

Adding slack variables, this LP can be solved to obtain the optimal solution

$$Z^* = x_5^* = 4, (x_1^*, x_2^*, x_3^*, x_4^*) = \left(0, \frac{7}{12}, \frac{5}{12}, 0 \right)$$

and the dual optimal solution (coefficients for slack variables)

$$W^* = y_5^* = 4, \quad (y_1^*, y_2^*, y_3^*, y_4^*) = \left(0, \frac{7}{12}, \frac{5}{12}, 0\right).$$

Note x^* identifies the optimal strategy for row player, and y^* identifies the optimal strategy for the column player. In this case they are the same because the game is symmetric. And as expected, the value of the game is 4 (therefore, the value for the original Two-Finger Morra is again $4 - 4 = 0$).

Question 1: Suppose that the row player play optimally using mixed strategy x^* , and the column player selects an arbitrary mixed strategy y . What is the expected loss for the column player?

Answer: The expected loss for the column player is

$$\sum_{i=1}^4 \sum_{j=1}^4 x_i^* b_{ij} y_j = \sum_{j=1}^4 y_j \left(\sum_{i=1}^4 x_i^* b_{ij} \right) = \frac{1}{12} y_1 + \frac{1}{12} y_2 \geq 0.$$

If the column player chooses a mixed strategy other than the optimal strategy y^* , he/she will not benefit from this change. In particular, if he chooses a mixed strategy with either $y_1 > 0$ or $y_2 > 0$, he/she will do worse. This confirms that (x^*, y^*) is an equilibrium point.

Question 2: This game indeed has more than one saddle point. Verify that for any $0 < a < 1$, the mixed strategies

$$\tilde{x} = \tilde{y} = \left(0, \frac{3a}{5} + \frac{4(1-a)}{7}, \frac{2a}{5} + \frac{3(1-a)}{7}, 0\right)$$

are optimal for the row player and column player, respectively.

Solution: Recall the theorem that states that if one can find a \tilde{v} such that

$$\sum_{i=1}^m \tilde{x}_i b_{ij} \geq \tilde{v} \quad \text{for all } j \quad \text{and} \quad \sum_{j=1}^n \tilde{y}_j b_{ij} \leq \tilde{v} \quad \text{for all } i$$

then \tilde{v} is the value of the game, and (\tilde{x}, \tilde{y}) is a saddle point.

In this case, we will choose $\tilde{v} = 0$, and we can verify that

$$\sum_{i=1}^4 \tilde{x}_i b_{i1} = \frac{1-a}{7}, \quad \sum_{i=1}^4 \tilde{x}_i b_{i2} = 0, \quad \sum_{i=1}^4 \tilde{x}_i b_{i3} = 0, \quad \sum_{i=1}^4 \tilde{x}_i b_{i4} = \frac{a}{5}.$$

They are all bigger than or equal to $\tilde{v} = 0$. We can similarly verify \tilde{y} . Therefore, (\tilde{x}, \tilde{y}) is also a saddle point.

Exercise: Solve the game with payoff matrix (i.e. identify value of the game and the saddle point)

$$B = \begin{bmatrix} 5 & -7 \\ -9 & 4 \end{bmatrix}$$

Exercise: Consider a hide-and-seek game with matrix

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Write down the payoff matrix and write down the LP to solve the game.

2 Two-person, Non-zero-sum games

The two-person, non-zero-sum game is much more complicated than the zero-sum game, as we have already seen in the prisoner's dilemma. Mathematically, in the zero-sum game, maximizing a player's own reward is equivalent to minimizing its opponent's payoff, which is not necessarily the case in a non-zero sum game.

The payoff matrix will usually denoted by two matrices, one for the row player and one for the column player

$$B_R = [r_{ij}]_{m \times n} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix}, \quad B_C = [c_{ij}]_{m \times n} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

The zero-sum game is just a special case where $B_R + B_C = 0$.

Definition: A strategy pair (i^*, j^*) is said to be a *pure-strategy Nash equilibrium* if

$$r_{ij^*} \leq r_{i^*j^*}, \quad \forall i = 1, 2, \dots, m \quad \text{and} \quad c_{i^*j} \leq c_{i^*j^*} \quad \forall j = 1, 2, \dots, n.$$

The pair $(r_{i^*j^*}, c_{i^*j^*})$ is said to be a *Nash equilibrium outcome*.

Parallel to the zero-sum game, a pure-strategy Nash equilibrium may fail to exist for arbitrary payoff matrices (B_R, B_C) . However, mixed strategy will again save the day.

Definition: A mixed-strategy pair (x^*, y^*) is said to be a *Nash equilibrium* if

$$\sum_{i=1}^m \sum_{j=1}^n r_{ij} x_i^* y_j^* \geq \sum_{i=1}^m \sum_{j=1}^n r_{ij} x_i y_j^*, \quad \text{for any mixed strategy } x,$$

and

$$\sum_{i=1}^m \sum_{j=1}^n c_{ij} x_i^* y_j^* \geq \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_i^* y_j, \quad \text{for any mixed strategy } y.$$

The pair $(\sum_i \sum_j r_{ij} x_i^* y_j^*, \sum_i \sum_j c_{ij} x_i^* y_j^*)$ is said to be a *Nash equilibrium outcome*.

Theorem: Nash equilibrium always exists.

The proof of this theorem utilizes a deep result (Kakutani's Fixed Point Theorem), and we will omit the proof.

Example: Suppose that

$$B_R = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}, \quad B_C = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$$

It is not difficult to show that there is no pure-strategy Nash equilibrium. However, $x^* = (0.5, 0.5)$, $y^* = (0.5, 0.5)$ is a mixed-strategy Nash equilibrium.

Remark: When a non-zero sum game has multiple equilibria, things can get complicated. Consider the example

$$B_R = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}, \quad B_C = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$$

It is not difficult to check that $(1, 1)$ and $(2, 2)$ are both pure-strategy Nash equilibria. The first saddle point will yield an outcome $(2, 1)$ and the second saddle point will yield an outcome $(1, 2)$. What should the players do?