

Game Theory

We will focus mainly on *two-person, zero-sum* games in this chapter. Game of this type involves two players, and one player wins whatever the other player loses (so the sum of their net winning is always zero). In the now classical book *Theory of Games and Economic Behavior*, John von Neumann proved an important theorem, i.e. *every two-person zero-sum game has a value*. His proof uses the Brouwer fixed-point theorem. We will present a proof using linear programming dual theory. The latter is superior because it is constructive – it shows you how to construct an optimal strategy for each player.

1 The formulation of two-person games

There are two players in this type of games, each has several *strategies* (or, *actions*) in disposal. Each player will pick a strategy and obtain the corresponding payoff, which is recorded in the **payoff matrix** (or *payoff table*). We will always assume that the game is *non-cooperative*, that is, each player has no sympathy for the opponent and chooses strategies solely to promote his own welfare.

Example (Non-zero-sum game): One of the most famous games is called the *Prisoner's dilemma*.

Two prisoners who escaped and participated in a robbery have been captured and are waiting for the trial for their new crime. Although they are both guilty, the district attorney is not sure he has enough evidence to convict them. To entice them to testify against each other, the D.A. tells each prisoner separately that: “If only one of you confesses and testifies against your partner, the person who confesses will go free while the person who does not confess will be convicted and given 20 years in jail. If you both confess, you will both be convicted and sent to prison for 5 year. If none of you confess, I can convict you both of misdemeanor and you will each get 1 year in prison.” What should the prisoners do, if they cannot communicate?

Solution: Each prisoner has two (pure) strategies to take: confess or not. We have the following payoff matrix.

		Prisoner B	
		Confess	Don't Confess
Prisoner A	Confess	(5,5)	(0,20)
	Don't Confess	(20,0)	(1,1)

Terminologies: Prisoner A is called the **row player** and prisoner B is called the **column player**.

For the reward in each cell, the first number is the years prisoner A (row player) will receive, and the second number is the years prisoner B (column player) will receive. It is important to note that the summation of the reward in each cell ranges from 2 to 20. The game is therefore a non-zero-sum game. Note that a zero-sum game requires the summation in each cell to be zero.

We will introduce some terminologies for future reference. To each prisoner, the strategy “don’t confess” is always dominated by the “confess” strategy, regardless of what the opponent (the other prisoner) will do. For example, no matter what prisoner B does, it is better off for prisoner A to confess ($5 < 20$ and $0 < 1$). More generally,

Definition: we say a strategy is **dominated** by a second strategy if the second strategy is *always at least as good* regardless of what the opponent does. A strategy is **undominated** if it is not dominated by any other strategy. A strategy is said to be **dominant** if it dominates any other strategies.

If each prisoner follows its dominant strategy “confess”, the outcome will be that each prisoner receives 5 years. On the other hand, if each prisoner follows its dominated strategy “don’t confess”, each will only receive 1 year.

This example illustrates an important aspect of Prisoner’s Dilemma type of game: If the prisoners are cooperating (if each prisoner chooses “don’t confess”), each prisoner can gain by double-crossing his opponent (assume his opponent’s strategy remains unchanged). If both prisoners double-cross each other, however, they will both be worse off than if they had both chosen their cooperative strategy.

Later we will show that (“confess”, “confess”) is the *Nash equilibrium*.

Example (zero-sum game): One of the simplest two-person zero-sum game is as follows. Two players (called Odd and Even) simultaneously stick out 1 or 2 fingers. If the sum of the fingers put out by the two players is odd, Odd wins one dollar from Even. Otherwise, Even wins one dollar from Odd. If we consider Odd as the row player and Even as the column player, the payoff matrix for this game is

		Even	
		1 finger	2 fingers
Odd	1 finger	(−1, 1)	(1, −1)
	2 fingers	(1, −1)	(−1, 1)

Note that in each cell the reward to Odd and the reward to Even always add up to zero (i.e. *zero sum*).

- By convention, for a zero-sum game, we only write down the payoff matrix for the row player, i.e.

		Even	
		1 finger	2 fingers
Odd	1 finger	−1	1
	2 fingers	1	−1

In this game, there is no dominant strategy for either player (why?). What should each player do?

Example (zero-sum game): In this example, we illustrate the possibility that one player has a dominant strategy while the other does not, using a famous game actually played in a naval engagement in 1943, during World War II. The game in question, the *battle of the Bismarck Sea* is named for the body of water in southwestern Pacific Ocean separating the Bismarck Archipelago from Papua-New Guinea.

In 1943, a Japanese admiral was ordered to transport troops and lead a convoy to New Guinea. The Japanese had two choices: a rainy northern route or a sunnier southern route. The U.S. airforces knew that the convoy would sail and wanted to send bombers after it. But they did not know which route the Japanese would take. The Americans had to send reconnaissance aircraft to scout for the Japanese, but they had only enough planes to explore one route at a time.

The sailing time was three days for both route. If the Japanese convoy was on the route that the Americans explored first, the U.S. could send its bombers straightaway; if not, a day of bombing was lost by the Americans. In addition, the poor weather on the northern route made it likely that visibility would be too limited for bombing on one day in three. Thus the Americans could anticipate two days of active bombing if they explored the northern route and found the Japanese immediately, and two days of bombing if they explored the northern route but discovered that the Japanese had gone south. If the Americans explore the southern route first and found the Japanese there, they could get in three days of bombing, but if they found the Japanese had gone north, they would only get one day of active bombing.

The can be looked upon as a zero-sum game. The payoff table for this game is as follows.

		Japanese Navy	
		North	South
U.S. Airforces	North	2	2
	South	1	3

The U.S. does better by choosing North if the Japanese has chosen North (two days of bombing rather than one), but it does better choosing South if the Japanese has chosen South (three days of bombing rather than two). The Americans, therefore, have no dominant strategy. But the Japanese do. No matter what Americans do, the Japanese navy is better off choosing the northern route. So North is the dominant strategy for the Japanese.

The choice for Japan is clear now – they will choose the northern route. The Americans can now take this into account when making their own decision. The U.S. airforces should choose the best strategy with the expectation that the Japanese will play North. That means that the U.S. will send their reconnaissance to the north. Each side goes north, and the Americans get two days of active bombing. Indeed, (North, North) is the Nash-equilibrium for this game.

It is not known whether the commanders on the two sides in 1943 thought thing this way. It is known, however, the outcome was just as the game theory predicts. The Japanese convoy took the northern route and American planes also searched there, and two days of bombing inflicted serious damage on the Japanese ships.

Example (zero-sum game): The game *hide and seek* was considered by von Neumann: Given a matrix

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Player A picks a row or a column, while player B picks a single component. Suppose player B picks B_{ij} , then player B must pay player A the amount of B_{ij} if player A picks either row i or column j (B hides, A seeks). But the payoff is zero when player A picks a row or a column not containing B_{ij} .

This is a zero-sum game and the payoff matrix is 5 by 6.

		Player B					
		B_{11}	B_{12}	B_{13}	B_{21}	B_{22}	B_{23}
Player A	Row 1	1	2	3	0	0	0
	Row 2	0	0	0	4	5	6
	Column 1	1	0	0	4	0	0
	Column 2	0	2	0	0	5	0
	Column 3	0	0	3	0	0	6

What should each player do? Is there any dominant strategy?

2 Solving simple games, pure strategies, saddle point

A **pure strategy** specifies a non-random course of action for a player; that is, the move to be made is specified without any uncertainty. For the *Battle of the Bismarck Sea* game, it turns out that the best strategy for the American is to explore the northern route. We will study pure strategies in zero-sum games and the important concept of **saddle point**.

Example (zero-sum game, saddle point): During the 8 to 9 P.M. time slot, two networks are vying for audience of 100 million viewers. The network must simultaneously announce the type of show they will air in that time slot. The possible choices for each network and the number of viewer for each choice is shown in the following table.

		Network 2		
		Western	Soap Opera	Comedy
Network 1	Western	(35, 65)	(15, 85)	(60, 40)
	Soap Opera	(45, 55)	(58, 42)	(50, 50)
	Comedy	(38, 62)	(14, 86)	(70, 30)

Even though the summation of rewards in each cell is 100 million, the game can be regarded as a zero-sum game, since the audience one network gains is the audience the other network loses. We can simplify the payoff table as

		Network 2		
		Western	Soap Opera	Comedy
Network 1	Western	35	15	60
	Soap Opera	45	58	50
	Comedy	38	14	70

with the convention that each cell is the reward the row player gets. It is easy to see in this case there is no dominant or dominated strategy.

How should the row player, network 1, play this game? It has three pure strategies:

1. Suppose network 1 selects the pure strategy “Western”. The column player, network 1, should choose “Soap Opera” to minimize the row player’s reward. In other words, network 1 should expect to receive (in the worst case) a reward of equal the *minimal reward in the row of “Western”*, which is 15.
2. Suppose network 1 selects the pure strategy “Soap Opera”. Similarly, network 1 should expect to receive (in the worst case) a reward of equal the *minimal reward in the row of “Soap Opera”*, which is 45.
3. Suppose network 1 selects the pure strategy “Comedy”. Similarly, network 1 should expect to receive (in the worst case) a reward of equal the *minimal reward in the row of “Comedy”*, which is 14.
4. Since network 1 tries to maximize the reward, it will choose the pure strategy “Soap Opera”, so that no matter what network 2 does, network 1 will receives at least a reward as 45. In other words, the row player chooses a strategy so as to receive a reward at least

$$\max \{\text{row minimum} : \text{all rows}\}$$

How should the column player, network 2, play this game? It has three pure strategies:

1. Suppose network 2 selects the pure strategy “Western”. The row player, network 1, should choose “Soap Opera” to maximize its own reward. In other words, network 2 should expect to lose (in the worst case) audience of size equal the *maximum in the column of “Western”*, which is 45.
2. Suppose network 2 selects the pure strategy “Soap Opera”. Similarly, network 2 should expect to lose (in the worst case) audience of size equal the *maximum in the column of “Soap Opera”*, which is 58.
3. Suppose network 2 selects the pure strategy “Comedy”. Similarly, network 2 should expect to lose (in the worst case) audience of size equal the *maximum in the column of “Comedy”*, which is 70.
4. Since network 2 tries to maximize the reward or minimize the loss, it will choose the pure strategy “Western”, so that no matter what network 1 does, network 2 will at most lose audience of size 45. In other words, the column player chooses a strategy so as to only lose by

$$\min \{\text{column maximum} : \text{all columns}\}$$

Conclusion: For this example, the game satisfies

$$(*) \quad \max \{ \text{row minimum} : \text{all rows} \} = \min \{ \text{column maximum} : \text{all columns} \},$$

which are both equal to 45. Condition (*) is called the **minmax criterion**. The strategy for network 1 is “Soap Opera” and the strategy for network 2 is “Western”. The pair of strategy (“Soap Opera”, “Western”) is called a **saddle point** (or, **equilibrium point**). Note this point simultaneously achieves the minimum of the row and the maximum of the column. The saddle point has a very important property that *neither player can benefit from a unilateral change in strategy* (why?). \square

2.1 General discussion

Consider a zero-sum game with a general payoff matrix

$$B = [b_{ij}]_{m \times n} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix},$$

with the understanding that the row player has m strategies, the column player has n strategies, and the payoff to the row player is b_{ij} and the payoff to the column player is $-b_{ij}$ when the row player chooses the i -th strategy and the column chooses the j -th strategy.

Definition: We say that the game with payoff matrix B satisfies the **minmax condition** if

$$\max_i \left(\min_j b_{ij} \right) = \min_j \left(\max_i b_{ij} \right) := v.$$

Suppose the minmax condition holds. The value v is then said to be the **value of the game**. If we further denote by i^* (resp. j^*) the row (resp. column) achieving the maximum (resp. minimum) in the left-hand-side (resp. right-hand-side), the pair of strategies (i^*, j^*) is said to be a **pure-strategy saddle point** or simply a **saddle point**.

Theorem: Assume the minmax condition holds. Then $b_{i^*j^*} = v$ and

$$(1) \quad b_{ij^*} \leq b_{i^*j^*} \leq b_{i^*j}, \quad \forall 1 \leq i \leq m, 1 \leq j \leq n.$$

The reverse is also true; i.e. if (1) holds for some pair (i^*, j^*) , then the game satisfies the minmax condition, (i^*, j^*) is a saddle point and the value of the game v equals $b_{i^*j^*}$.

Remark: The inequality (1) says that the saddle point can be thought of as an *equilibrium point* in the sense that neither player can benefit from a unilateral change in strategy.

Proof: Assume the minmax condition holds. It follows that

$$v = \left(\min_j b_{i^*j} \right) = \left(\max_i b_{ij^*} \right).$$

In particular, we have

$$b_{i^*j^*} \leq \left(\max_i b_{ij^*} \right) = v = \left(\min_j b_{i^*j} \right) \leq b_{i^*j^*},$$

from which we conclude $b_{i^*j^*} = v$, which in turn implies that

$$b_{i^*j^*} = \left(\max_i b_{ij^*} \right) \geq b_{ij^*}, \quad \text{for all } i, \quad \text{and} \quad b_{i^*j^*} = \left(\min_j b_{i^*j} \right) \leq b_{i^*j}, \quad \text{for all } j.$$

For the reverse direction, assume that (1) holds. We have

$$\left(\max_i b_{ij^*} \right) = b_{i^*j^*} = \left(\min_j b_{i^*j} \right),$$

which further implies

$$\max_i \left(\min_j b_{ij} \right) \geq \left(\min_j b_{i^*j} \right) = b_{i^*j^*} = \left(\max_i b_{ij^*} \right) \geq \min_j \left(\max_i b_{ij} \right).$$

It follows from the lemma below that

$$\min_j \left(\max_i b_{ij} \right) = \max_i \left(\min_j b_{ij} \right),$$

and thus

$$\max_i \left(\min_j b_{ij} \right) = \left(\min_j b_{i^*j} \right) = b_{i^*j^*} = \left(\max_i b_{ij^*} \right) = \min_j \left(\max_i b_{ij} \right)$$

or the minmax condition holds. Clearly i^* achieves the maximum of the left-hand-side and j^* achieves the minimum of the right-hand-side; i.e., (i^*, j^*) is a saddle point. \square

Lemma: For any payoff matrix B , we have

$$\max_i \left(\min_j b_{ij} \right) \leq \min_j \left(\max_i b_{ij} \right)$$

Proof: It suffices to prove that

$$\min_j b_{ij} \leq \min_j \left(\max_i b_{ij} \right), \quad \text{for all } i.$$

However, this is trivial since

$$b_{ij} \leq \max_i b_{ij}, \quad \text{for all } j.$$

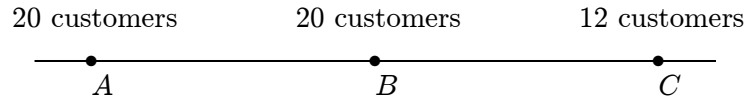
We complete the proof. \square

Example: Check that the *Battle of the Bismarck Sea* game satisfies the minmax condition, and (“North”, “North”) is a saddle point.

Example: Check that the game of *Odd-Even* does not satisfy the minmax condition, hence the game has no pure-strategy saddle point.

Example: Check that the game of *hide and seek* does not satisfy the minimax condition, and hence has no pure-strategy saddle point.

Example: Two competing firms are deciding whether to locate a new store in village A , B , or C . There are 52 perspective customers for the two stores. 20 customers live in village A , 20 customers live in village B , and 12 customers live in village C . Each customer will shop at the nearer store. If the customer is equidistant from both stores, it is assumed that there is a 50% chance that he or she will shop at either store. Each firm want to maximize the expected number of customers that will shop at its store. Where should each firm locate its store? ($AB = BC = 10$ miles)



Solution: This game can be viewed as a zero sum game as in the network game. The following payoff matrix is the expected number of customers for Firm 1.

		Firm 2			Row min
		A	B	C	
Firm 1	A	26	20	30	20
	B	32	26*	40	26
	C	22	12	26	12
Column max		32	26	40	$v = 26$

That is, both firms should build the store at village B . □

3 Mixed strategies

As we have seen, a two-player zero-sum game may not have a pure-strategy saddle point. If this is the case, it seems impossible to determine what is the best strategy for either player. To overcome this difficulty, we now introduce the concept of mixed strategies.

Definition: A **mixed strategy** specifies that the actual move be chosen randomly from the set of pure strategies with some specific probabilities. For example, suppose a player has n pure strategies. Then for any vector (x_1, x_2, \dots, x_n) such that $x_i \geq 0$ and $x_1 + x_2 + \dots + x_n = 1$ defines a mixed strategy by

$$x_i = \text{the probability that the player chooses } i\text{-th strategy}; \quad \forall i = 1, 2, \dots, n$$

A pure strategy, say the i -th strategy, can be viewed as a special mixed strategy with $x_i = 1$ and $x_j = 0$ for all $j \neq i$.

The importance of mixed strategies is that any two-person zero-sum game has a value and a Nash equilibrium point in mixed strategies.

Example: Consider the Odd-Even game. Suppose that now player Odd chooses a mixed strategy (x_1, x_2) with $x_1, x_2 \geq 0$, and $x_1 + x_2 = 1$. What is the worst case for him? Since the expected payoff for Odd is $x_2 - x_1 = 1 - 2x_1$ if Even chooses strategy “1 finger” and $x_1 - x_2 = 2x_1 - 1$ if Even chooses strategy “2 fingers”, the expected payoff for player Odd would be

$$y_1(1 - 2x_1) + y_2(2x_1 - 1).$$

if player Even chooses a mixed strategy (y_1, y_2) with $y_1, y_2 \geq 0$, and $y_1 + y_2 = 1$. Player Even will try to minimize the payoff for player Odd. Therefore the worst case for Odd is

$$\min\{1 - 2x_1, 2x_1 - 1\}.$$

Therefore, player Odd should choose a strategy (x_1, x_2) so as to maximize the above quantity. This can easily be done by graph, and

$$\max_{0 \leq x_1 \leq 1} \min\{1 - 2x_1, 2x_1 - 1\} = 0, \quad \text{achieved at } x_1^* = \frac{1}{2}.$$

Therefore, the expected payoff for player Odd is at least 0 if he chooses a mixed strategy $(0.5, 0.5)$.

What about player Even? Suppose Even chooses a mixed strategy (y_1, y_2) with $y_1, y_2 \geq 0$, and $y_1 + y_2 = 1$. What is the worst case for him? Since the expected loss for Even is $-y_1 + y_2 = 1 - 2y_1$ if Odd chooses strategy “1 finger” and $y_1 - y_2 = 2y_1 - 1$ if Odd chooses strategy “2 fingers”, the expected loss for player Even is

$$x_1(1 - 2y_1) + x_2(2y_1 - 1).$$

if player Odd chooses a mixed strategy (x_1, x_2) with $x_1, x_2 \geq 0$, and $x_1 + x_2 = 1$. Player Odd will try to maximize its own payoff. Therefore the worst case for Even is to lose

$$\max\{1 - 2y_1, 2y_1 - 1\}.$$

Therefore, player Even should choose a strategy (y_1, y_2) so as to minimize the above quantity. This can easily be done by graph, and

$$\min_{0 \leq y_1 \leq 1} \max\{1 - 2y_1, 2y_1 - 1\} = 0, \quad \text{achieved at } y_1^* = \frac{1}{2}.$$

Therefore, the expected loss for player Even is at most 0 if he chooses a mixed strategy $(0.5, 0.5)$.

The game of Odd-Even, hence, has a value 0, and the saddle point for the game is that both player chooses a mixed strategy $(0.5, 0.5)$. Neither player can benefit from a unilateral change in strategy and will face the risk of getting an expected reward less than the value of the game 0. For example, if player Odd chooses a strategy (x_1, x_2) with $x_1 > 0.5$, he will get an expected reward $-x_1 + x_2 = 1 - 2x_1 < 0$ if Even chooses a pure strategy of “1 finger”.

3.1 General discussion on mixed strategies

We will assume the payoff for the row player is matrix $B = [b_{ij}]_{m \times n}$ as before.

Suppose the row player chooses a mixed strategy $x = (x_1, x_2, \dots, x_m)$ such that $x \geq 0$ and $\sum_{i=1}^m x_i = 1$. The column player should choose a mixed strategy $y = (y_1, y_2, \dots, y_n)$ such that $y \geq 0$ and $\sum_{j=1}^n y_j = 1$, so as to minimize the expected payoff for the row player. In other words, the column player will try to

$$\text{Minimize} \quad \sum_{i=1}^m \sum_{j=1}^n x_i y_j b_{ij} = \sum_{j=1}^n y_j \left(\sum_{i=1}^m x_i b_{ij} \right).$$

This minimization problem is trivial – since $\sum_{j=1}^n y_j = 1$ and $y_j \geq 0$, the minimum is indeed

$$\min_{1 \leq j \leq n} \left(\sum_{i=1}^m x_i b_{ij} \right).$$

Or the row player will receive the above amount of expected payoff in the worst possible case. Therefore, the row player will try to pick a mixed strategy x so as to maximize the above minimum, or equivalently

$$\underline{v} \doteq \max_x \min_y \left(\sum_{i=1}^m \sum_{j=1}^n x_i y_j b_{ij} \right) = \max_x \min_{1 \leq j \leq n} \left(\sum_{i=1}^m x_i b_{ij} \right).$$

The value \underline{v} is said to be the *lower value of the game*.

Similarly, suppose the column player choose a mixed strategy y . The row player should choose a mixed strategy x so as to maximize its expected reward

$$\text{Maximize} \quad \sum_{i=1}^m \sum_{j=1}^n x_i y_j b_{ij}.$$

The column player will chooses a mixed strategy y to minimize the expected loss in the worst case (i.e., the above maximum). Define

$$\bar{v} \doteq \min_y \max_x \left(\sum_{i=1}^m \sum_{j=1}^n x_i y_j b_{ij} \right) = \min_y \max_{1 \leq i \leq m} \left(\sum_{j=1}^n y_j b_{ij} \right).$$

The value \bar{v} is said to be the *upper value of the game*.

We have the following important result.

Theorem 1: The lower value of the game coincides with the upper value of the game. We call $\underline{v} = \bar{v} \doteq v$ the **value of the game**. Furthermore, suppose x^* achieves the maximum in the definition of \underline{v} and y^* achieves the minimum in the definition of \bar{v} . Then we have

$$\sum_{i=1}^m \sum_{j=1}^n x_i y_j^* b_{ij} \leq \sum_{i=1}^m \sum_{j=1}^n x_i^* y_j^* b_{ij} = v \leq \sum_{i=1}^m \sum_{j=1}^n x_i^* y_j b_{ij}$$

for any mixed strategy x for row player and any mixed strategy y for column player. In particular,

$$\sum_{j=1}^n x_i^* b_{ij} \geq v \quad \text{for all } j = 1, 2, \dots, n.$$

and

$$\sum_{j=1}^n y_j^* b_{ij} \leq v \quad \text{for all } i = 1, 2, \dots, m$$

The pair of strategies (x^*, y^*) is called an **saddle point**,

Remark: If the row player adopts the saddle point strategy x^* , its expect reward is at least v , regardless of the strategy its opponent picks. Similarly, if the column player chooses the saddle point strategy y^* , its expect loss is at most v , regardless of the strategy its opponent picks. Neither player benefit from a unilateral change in strategy. For this reason, the saddle point is also called the *equilibrium*, and x^* (resp. y^*) is called the **optimal strategy** for the row (resp. column) player.

Theorem 2: Suppose there is a real number \tilde{v} , a mixed strategy \tilde{x} for the row player, and a mixed strategy \tilde{y} for the column player, such that

$$\sum_{i=1}^m \tilde{x}_i b_{ij} \geq \tilde{v} \quad \text{for all } j = 1, 2, \dots, n.$$

and

$$\sum_{j=1}^n \tilde{y}_j b_{ij} \leq \tilde{v} \quad \text{for all } i = 1, 2, \dots, m$$

Then $\tilde{v} = v$ is the value of the game, and (\tilde{x}, \tilde{y}) is a saddle point.

The proof used the dual theory for linear programming, and is constructive – it also tells us how to find the saddle point.

Proof of Theorem 1: Let us first consider the lower value of the game \underline{v} , which is a maximization problem

$$\max_x \min_{1 \leq j \leq n} \left(\sum_{i=1}^m x_i b_{ij} \right).$$

This can be written as a LP problem: write $x_{m+1} = \min_{1 \leq j \leq n} (\sum_{i=1}^m x_i b_{ij})$. The LP is

$$\text{Maximize } Z = x_{m+1}$$

under constraints

$$\begin{array}{rclclcl} -x_1 b_{11} & - & x_2 b_{21} & - & \cdots & - & x_m b_{m1} & + & x_{m+1} & \leq & 0 \\ -x_1 b_{12} & - & x_2 b_{22} & - & \cdots & - & x_m b_{m2} & + & x_{m+1} & \leq & 0 \\ & & & & & & & & & & \vdots \\ -x_1 b_{1n} & - & x_2 b_{2n} & - & \cdots & - & x_m b_{mn} & + & x_{m+1} & \leq & 0 \\ x_1 & + & x_2 & + & \cdots & + & x_m & & & = & 1 \end{array}$$

and $x_1, \dots, x_m \geq 0$, while x_{m+1} has no sign constraints.

Its dual LP is

$$\begin{aligned}
& \text{Minimize } W = y_{n+1} \\
& \text{under constraints} \\
& \begin{aligned}
& -b_{11}y_1 - b_{12}y_2 - \cdots - b_{1n}y_n + y_{n+1} \geq 0 \\
& -b_{21}y_1 - b_{22}y_2 - \cdots - b_{2n}y_n + y_{n+1} \geq 0 \\
& \vdots \\
& -b_{m1}y_1 - b_{m2}y_2 - \cdots - b_{mn}y_n + y_{n+1} \geq 0 \\
& y_1 + y_2 + \cdots + y_n = 1
\end{aligned}
\end{aligned}$$

and $y_1, \dots, y_n \geq 0$, while y_{n+1} has no sign constraints.

This is indeed the LP corresponding to the minimization problem for \bar{v} :

$$\min_y \max_{1 \leq i \leq m} \left(\sum_{j=1}^n y_j b_{ij} \right).$$

Clearly, the primal LP and dual LP are both feasible. Therefore, we have

$$\bar{v} = \underline{v} \doteq v.$$

Furthermore, we have for every strategy y ,

$$v = \underline{v} = \max_{1 \leq j \leq n} \left(\sum_{i=1}^m x_i^* b_{ij} \right) \geq \sum_{j=1}^n y_j \left(\sum_{i=1}^m x_i^* b_{ij} \right) = \sum_{i=1}^m \sum_{j=1}^n x_i^* y_j b_{ij}.$$

and for every strategy x ,

$$v = \bar{v} = \min_{1 \leq i \leq m} \left(\sum_{j=1}^n y_j^* b_{ij} \right) \leq \sum_{i=1}^m x_i \left(\sum_{j=1}^n y_j^* b_{ij} \right) = \sum_{i=1}^m \sum_{j=1}^n x_i y_j^* b_{ij}.$$

In particular, we have

$$\sum_{i=1}^m \sum_{j=1}^n x_i^* y_j^* b_{ij} \leq v \leq \sum_{i=1}^m \sum_{j=1}^n x_i^* y_j^* b_{ij}.$$

This completes the proof. \square

Remark: The proof of Theorem 1 also indicates that finding the optimal strategy for the players is equivalent to solving the two LPs.

Proof of Theorem 2: By assumption, we have

$$\begin{aligned}
v = \bar{v} &= \min_y \max_{1 \leq i \leq m} \left(\sum_{j=1}^n \tilde{y}_j b_{ij} \right) \leq \max_{1 \leq i \leq m} \left(\sum_{j=1}^n \tilde{y}_j b_{ij} \right) \\
&\leq \tilde{v} \\
&\leq \min_{1 \leq j \leq n} \left(\sum_{i=1}^m \tilde{x}_i b_{ij} \right) \leq \max_x \min_{1 \leq j \leq n} \left(\sum_{i=1}^m \tilde{x}_i b_{ij} \right) = \underline{v} = v.
\end{aligned}$$

Therefore, all the inequalities are indeed equalities. We complete the proof. \square

Connection with pure strategoes: Previously, we consider pure strategies and define the value of a game when the game satisfies the minimax condition

$$\max_i \left(\min_j b_{ij} \right) = \min_j \left(\max_i b_{ij} \right) := \text{value of game},$$

and also defines the pure-strategy saddle point, say (i^*, j^*) . A question naturally arise: if the minimax condition holds, is there any contradiction to the value of game and saddle point obtained from mixed strategy approach?

The answer is “No”. Indeed, when the minimax condition holds, the value of game and saddle point from pure strategies are equivalent to those from mixed-strategies.

Proposition: Suppose the minimax condition holds. Then

$$\max_i \left(\min_j b_{ij} \right) = \min_j \left(\max_i b_{ij} \right) = \text{value of game from mixed strategy}.$$

Define \tilde{x} such that $\tilde{x}_{i^*} = 1$, $\tilde{x}_i = 0$ if $i \neq i^*$, and \tilde{y} such that $\tilde{y}_{j^*} = 1$, $\tilde{y}_j = 0$ if $j \neq j^*$. Then (\tilde{x}, \tilde{y}) is a saddle point from mixed strategy.

Proof: Let $\tilde{v} = \max_i (\min_j b_{ij}) = \min_j (\max_i b_{ij})$. For every $j = 1, \dots, n$, we have

$$\sum_{i=1}^m \tilde{x}_i b_{ij} = b_{i^*j} \geq b_{i^*j^*} = \tilde{v}.$$

Similarly,

$$\sum_{j=1}^n \tilde{y}_j b_{ij} = b_{ij^*} \leq b_{i^*j^*} = \tilde{v}.$$

We complete the proof, thanks to Theorem 2. □

Symmtric games: A two-person zero-sum game is said to be *symmetric* if the payoff matrix B satisfies

$$B = -B^T.$$

In particular, this implies $m = n$.

Proposition: We will show here that every symmetric zero-sum game has value 0, and there exists a saddle point (x^*, y^*) such that $x^* = y^*$.

Proof: Suppose the value of the game is v , then we have

$$\underline{v} = \max_x \min_{1 \leq j \leq n} \left(\sum_{i=1}^n x_i b_{ij} \right) = \max_x \min_{1 \leq j \leq n} \left(- \sum_{i=1}^n x_i b_{ji} \right) = - \min_x \max_{1 \leq j \leq n} \left(\sum_{i=1}^n x_i b_{ji} \right) = -\bar{v}.$$

Hence $v = \bar{v} = \underline{v} = 0$. Furthermore, suppose x^* achieves the maximum (0) in the definition of \underline{v} , it clearly achieves the minimum in the definition of \bar{v} . This completes the proof. □

4 Examples

From the previous section, we know that solving a game is as solving an LP. But for a simple game where each player only has two strategies, the solution is easy to obtain without resorting to LP.

Example: Suppose the payoff matrix is

$$B = \begin{bmatrix} 0 & a \\ c & 0 \end{bmatrix}$$

with $a, c > 0$. Compute the value of the game and the saddle point.

Solution: The row player will choose a mixed strategy so as to

$$\max_x \min_{1 \leq j \leq n} \left(\sum_{i=1}^m x_i b_{ij} \right) = \max_x \min (x_1 b_{11} + x_2 b_{21}, x_1 b_{12} + x_2 b_{22}) = \max_x \min (cx_2, ax_1)$$

Since $x_1 + x_2 = 1$, the maximizing x^* is such that $cx_2^* = ax_1^*$ or

$$x_1^* = \frac{c}{a+c}, \quad x_2^* = \frac{a}{a+c}.$$

The value of the game is therefore

$$v = \max_x \min (cx_2, ax_1) = \min (cx_2^*, ax_1^*) = \frac{ac}{a+c}.$$

Similarly, the column player will choose a mixed strategy (exercise!)

$$y_1^* = \frac{a}{a+c}, \quad y_2^* = \frac{c}{a+c}.$$

A bluffing game: Consider the following simple bluffing game. The first thing we both do is putting down a small positive ante a . Then you draw one card from an ordinary deck. After looking at it, you put it face down without showing it to me. We will say black cards are high, and red cards are low.

Here are the rules. After you draw, you may bet or fold. If you fold, you lose the ante a . If you bet, then I may fold or I may call. If I fold, I lose the ante a , whether you draw black and red. If I bet, then you win the amount b if you drew a black card, or I win the amount b if you drew a red card. (The ante a and bet size b are fixed in advance, with $0 < a < b$.)

Your pure strategies: If you draw black, you will certainly bet – there's no question about that; you will bet and win at least the ante a . The only question is this: Will you bet if you draw red? That would be the *bluff* strategy. If you fold when you draw red, that is the *fold* strategy. (Remember, if you are playing the fold strategy, you will still bet if you draw black).

My pure strategies: You have just bet. What should I do? If I know you only bet on black, I will fold; but if I think you may bluff and bet on red, I may decide to call you. I have two pure strategies: the *call* strategy, in which I will call if you bet; and the *fold* strategy, in which I will fold if you bet.

The payoff matrix for you is as follows:

		I	
		Call	Fold
You	Bluff	0	a
	Fold	$(b - a)/2$	0

Let us explain the four components.

- $b_{11} = 0$: If you draw black, you will bet, I will call, and you will win b . If you draw red, you will bluff, I will call, and you will lose b . Since black and red are equally likely, your *average* payoff is $(b - b)/2 = 0$.
- $b_{12} = a$: Whatever you draw, you will bet, I will fold, and you will win the ante a .
- $b_{21} = (b - a)/2$: If you draw black, you will bet, I will call, and you will win b . If you draw red, you will fold, and you will lose ante a . Your average payoff is $(b - a)/2$.
- $b_{22} = 0$: If you draw black, you will bet, I will fold, and you will win a . If you draw red, you will fold, and you will lose a . Your average payoff is 0.

What are the optimal strategies for the players? If you always play your *bluff* strategy, I will play my *call* strategy; If you always play *fold*, I will play *fold*. So, in pure strategies, I can hold your payoff down to zero.

It follows from the previous example that you have an optimal mixed strategy telling that you should play the bluff strategy with probability $(b - a)/(b + a)$ and the fold strategy with probability $(2a)/(a + b)$. The value of the game (your expected reward) is

$$v = a \frac{b - a}{b + a}.$$

Your optimal bluffing frequency depends on the bet-to-ante ratio $r = b/a$: the bigger r is, the more often you should bluff.

4.1 How to solve the LPs associated with the game

For more complicated payoff matrix B , we need to solve the LP of form:

$$\begin{aligned}
& \text{Maximize } Z = x_{m+1} \\
& \text{under constraints} \\
& \begin{array}{rclclcl}
-x_1 b_{11} & - & x_2 b_{21} & - & \cdots & - & x_m b_{m1} & + & x_{m+1} & \leq & 0 \\
-x_1 b_{12} & - & x_2 b_{22} & - & \cdots & - & x_m b_{m2} & + & x_{m+1} & \leq & 0 \\
& & & & & & \vdots & & & & \\
-x_1 b_{1n} & - & x_2 b_{2n} & - & \cdots & - & x_m b_{mn} & + & x_{m+1} & \leq & 0 \\
x_1 & + & x_2 & + & \cdots & + & x_m & & & = & 1
\end{array}
\end{aligned}$$

and $x_1, \dots, x_m \geq 0$, while x_{m+1} has no sign constraints.

One can certainly transform this LP into standard form, add slack and artificial variables, and use simplex algorithm. An easier approach is as follows: (a) Add a constant, say c , to the payoff

matrix so that every entry of the matrix is now non-negative. Of course, this will not change the optimal strategy whatsoever, but will raise the value of the game by c ; (b) Now we can assume $b_{ij} \geq 0$. And the original LP is equivalent to the LP in standard form (why?):

$$\begin{aligned}
& \text{Maximize } Z = x_{m+1} \\
& \text{under constraints} \\
& \begin{array}{rcl}
-x_1 b_{11} & - & x_2 b_{21} & - & \cdots & - & x_m b_{m1} & + & x_{m+1} & \leq & 0 \\
-x_1 b_{12} & - & x_2 b_{22} & - & \cdots & - & x_m b_{m2} & + & x_{m+1} & \leq & 0 \\
& \vdots & & & & & & & & & \\
-x_1 b_{1n} & - & x_2 b_{2n} & - & \cdots & - & x_m b_{mn} & + & x_{m+1} & \leq & 0 \\
& x_1 & + & x_2 & + & \cdots & + & x_m & & & \leq & 1
\end{array} \\
& \text{and } x_1, \dots, x_m \geq 0, x_{m+1} \geq 0.
\end{aligned}$$

We only need to add slack variables to this LP.

Exercise: Show that the dual of this new LP is equivalent to the dual of the original LP under the assumption that $b_{ij} \geq 0, \forall i, j$. This shows that when we obtain the optimal primal solution by simplex algorithm, we also obtain the optimal dual solution (coefficients for slack variables). Note the optimal primal (resp. dual) solution gives the optimal strategy for the row (resp. column) player.

Two Finger Morra: Two players in the game of Two-Finger Morra simultaneously put out either one or two fingers. Each player must also announce the number of fingers that he believes his opponent has put out. If neither or both players successfully guess the number of fingers put out by the opponent, the game is a draw. Otherwise, the player who guesses correctly wins from the other player the sum (in dollars) of the fingers put out by the two players.

The game is a zero-sum game. Let (i, j) be a strategy of putting out i fingers and guessing the opponent has put out j fingers. The payoff matrix is as follows.

		Column player				Row min
		(1,1)	(1,2)	(2,1)	(2,2)	
Row player	(1,1)	0	2	-3	0	-3
	(1,2)	-2	0	0	3	-2
	(2,1)	3	0	0	-4	-4
	(2,2)	0	-3	4	0	-3
Column max		3	2	4	3	

This game does not satisfy the minimax condition; i.e.

$$\max_i \left(\min_j b_{ij} \right) = -2 \neq 2 = \min_j \left(\max_i b_{ij} \right).$$

Therefore there is no pure-strategy saddle point.

However, zero-sum games always have a value when we consider mixed strategies. In this case, the game is symmetric; i.e. $B = -B^T$. Therefore the value of the game is 0. But what is the optimal strategy for each player? This can be solved by forming the equivalent LP.

First we make each entry in the payoff maxtrix non-negative by adding a constant, say 4. The payoff matrix is now

$$B = \begin{bmatrix} 4 & 6 & 1 & 4 \\ 2 & 4 & 4 & 7 \\ 7 & 4 & 4 & 0 \\ 4 & 1 & 8 & 4 \end{bmatrix}$$

The optimal strategy will not change by this operation, but the value will increase by 4. For the row player is to find a mixed strategy $x = (x_1, \dots, x_4)$ so as to

$$\max_x \min_y \left(\sum_{i=1}^4 \sum_{j=1}^4 x_i y_j b_{ij} \right) = \max_x \left(\sum_{j=1}^4 x_i b_{ij} \right).$$

The corresponding LP is

$$\text{Maximize } Z = x_5$$

under constraints

$$\begin{array}{rcccccccl} -4x_1 & - & 2x_2 & - & 7x_3 & - & 4x_4 & + & x_5 & \leq & 0 \\ -6x_1 & - & 4x_2 & - & 4x_3 & - & x_4 & + & x_5 & \leq & 0 \\ -x_1 & - & 4x_2 & - & 4x_3 & - & 8x_4 & + & x_5 & \leq & 0 \\ -4x_1 & - & 7x_2 & & & - & 4x_4 & + & x_5 & \leq & 0 \\ x_1 & + & x_2 & + & x_3 & + & x_4 & & & \leq & 1 \end{array}$$

and $x_1, \dots, x_4 \geq 0, x_5 \geq 0$.

Adding slack variables, this LP can be solved to obtain the optimal solution

$$Z^* = x_5^* = 4, (x_1^*, x_2^*, x_3^*, x_4^*) = (0, \frac{7}{12}, \frac{5}{12}, 0)$$

and the dual optimal solution (coefficients for slack variables)

$$W^* = y_5^* = 4, (y_1^*, y_2^*, y_3^*, y_4^*) = (0, \frac{7}{12}, \frac{5}{12}, 0).$$

Note x^* identifies the optimal strategy for row player, and y^* identifies the optimal strategy for the column player. In this case they are the same because the game is symmetric. And as expected, the value of the game is 4 (therefore, the value for the original Two-Finger Morra is again $4 - 4 = 0$).

Question 1: Suppose that the row player play optimally using mixed strategy x^* , and the column player selects an arbitrary mixed strategy y . What is the expected loss for the column player?

Answer: The expected loss for the column player is

$$\sum_{i=1}^4 \sum_{j=1}^4 x_i^* b_{ij} y_j = \sum_{j=1}^4 y_j \left(\sum_{i=1}^4 x_i^* b_{ij} \right) = \frac{1}{12} y_1 + \frac{1}{12} y_2 \geq 0.$$

If the column player chooses a mixed strategy other than the optimal strategy y^* , he/she will not benefit from this change. In particular, if he chooses a mixed strategy with either $y_1 > 0$ or $y_4 > 0$, he/she will do worse. This confirms that (x^*, y^*) is an equilibrium point.

Question 2: This game indeed has more than one saddle point. Verify that for any $0 < a < 1$, the mixed strategies

$$\tilde{x} = \tilde{y} = \left(0, \frac{3a}{5} + \frac{4(1-a)}{7}, \frac{2a}{5} + \frac{3(1-a)}{7}, 0 \right)$$

are optimal for the row player and column player, respectively.

Solution: Recall the theorem that states that if one can find a \tilde{v} such that

$$\sum_{i=1}^m \tilde{x}_i b_{ij} \geq \tilde{v} \quad \text{for all } j \quad \text{and} \quad \sum_{j=1}^n \tilde{y}_j b_{ij} \leq \tilde{v} \quad \text{for all } i$$

then \tilde{v} is the value of the game, and (\tilde{x}, \tilde{y}) is a saddle point.

In this case, we will choose $\tilde{v} = 0$, and we can verify that

$$\sum_{i=1}^4 \tilde{x}_i b_{i1} = \frac{1-a}{7}, \quad \sum_{i=1}^4 \tilde{x}_i b_{i2} = 0, \quad \sum_{i=1}^4 \tilde{x}_i b_{i3} = 0, \quad \sum_{i=1}^4 \tilde{x}_i b_{i4} = \frac{a}{5}.$$

They are all bigger than or equal to $\tilde{v} = 0$. We can similarly verify \tilde{y} . Therefore, (\tilde{x}, \tilde{y}) is also a saddle point.

Exercise: Solve the game with payoff matrix (i.e. identify value of the game and the saddle point)

$$B = \begin{bmatrix} 5 & -7 \\ -9 & 4 \end{bmatrix}$$

Exercise: Consider a hide-and-seek game with matrix

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Write down the payoff matrix and write down the LP to solve the game.

5 Two-person, Non-zero-sum games

The two-person, non-zero-sum game is much more complicated than the zero-sum game, as we have already seen in the prisoner's dilemma. Mathematically, in the zero-sum game, maximizing

a player's own reward is equivalent to minimizing its opponent's payoff, which is not necessarily the case in a non-zero sum game.

The payoff matrix will usually denoted by two matrices, one for the row player and one for the column player

$$B_R = [r_{ij}]_{m \times n} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix}, \quad B_C = [c_{ij}]_{m \times n} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

The zero-sum game is just a special case where $B_R + B_C = 0$.

Definition: A strategy pair (i^*, j^*) is said to be a *pure-strategy Nash equilibrium* if

$$r_{ij^*} \leq r_{i^*j^*}, \quad \forall i = 1, 2, \dots, m \quad \text{and} \quad c_{i^*j} \leq c_{i^*j^*} \quad \forall j = 1, 2, \dots, n.$$

The pair $(r_{i^*j^*}, c_{i^*j^*})$ is said to be a *Nash equilibrium outcome*.

Parallel to the zero-sum game, a pure-strategy Nash equilibrium may fail to exist for arbitrary payoff matrices (B_R, B_C) . However, mixed strategy will again save the day.

Definition: A mixed-strategy pair (x^*, y^*) is said to be a *Nash equilibrium* if

$$\sum_{i=1}^m \sum_{j=1}^n r_{ij} x_i^* y_j^* \geq \sum_{i=1}^m \sum_{j=1}^n r_{ij} x_i y_j^*, \quad \text{for any mixed strategy } x,$$

and

$$\sum_{i=1}^m \sum_{j=1}^n c_{ij} x_i^* y_j^* \geq \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_i^* y_j, \quad \text{for any mixed strategy } y.$$

The pair $(\sum_i \sum_j r_{ij} x_i^* y_j^*, \sum_i \sum_j c_{ij} x_i^* y_j^*)$ is said to be a *Nash equilibrium outcome*.

Theorem: Nash equilibrium always exists.

The proof of this theorem utilizes a deep result (Kakutani's Fixed Point Theorem), and we will omit the proof.

Example: Suppose that

$$B_R = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}, \quad B_C = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$$

It is not difficult to show that there is no pure-strategy Nash equilibrium. However, $x^* = (0.5, 0.5)$, $y^* = (0.5, 0.5)$ is a mixed-strategy Nash equilibrium.

Remark: When a non-zero sum game has multiple equilibria, things can get complicated. Consider the example

$$B_R = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}, \quad B_C = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$$

It is not difficult to check that $(1, 1)$ and $(2, 2)$ are both pure-strategy Nash equilibria. The first saddle point will yield an outcome $(2, 1)$ and the second saddle point will yield an outcome $(1, 2)$. What should the players do?